Invent. math. 149, 97–122 (2002)
Digital Object Identifier (DOI) 10.1007/s002220200214

Inventiones mathematicae

The Baum-Connes conjecture for hyperbolic groups

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Oblatum 20-VI-2001 & 24-VIII-2001
Published online: 15 April 2002 –

Abstract. We prove the Baum-Connes conjecture for hyperbolic groups and their subgroups.

1. Introduction

The Baum-Connes conjecture states that, for a discrete group G, the K-homology groups of the classifying space for proper G-action is isomorphic to the K-groups of the reduced group C^*-algebra of G [3,2]. A positive answer to the Baum-Connes conjecture would provide a complete solution to the problem of computing higher indices of elliptic operators on compact manifolds. The rational injectivity part of the Baum-Connes conjecture implies the Novikov conjecture on homotopy invariance of higher signatures. The Baum-Connes conjecture also implies the Kadison-Kaplansky conjecture that for G torsion free there exists no non-trivial projection in the reduced group C^*-algebra associated to G. In [7], Higson and Kasparov prove the Baum-Connes conjecture for groups acting properly and isometrically on a Hilbert space. In a recent remarkable work, Vincent Lafforgue proves the Baum-Connes conjecture for strongly bolic groups with property RD [15,12,13]. In particular, this implies the Baum-Connes conjecture for the fundamental groups of strictly negatively curved compact manifolds. In [4], Connes and Moscovici prove the rational injectivity part of the Baum-Connes conjecture for hyperbolic groups using cyclic cohomology method. In [11], Kasparov and Skandalis prove the rational injectivity of

⋆ The second author is partially supported by NSF and MSRI.

the Baum-Connes conjecture for bolic groups using KK-theory. In this paper, we exploit Lafforgue's work to prove the Baum-Connes conjecture for hyperbolic groups and their subgroups.

The main step in the proof is the following theorem.

Theorem 17. *Every hyperbolic group G admits a metric $\hat{d}$ with the following properties.*

(1) *$\hat{d}$ is G-invariant, i.e. $\hat{d}(g \cdot x, g \cdot y) = \hat{d}(x, y)$ for all $x, y, g \in G$.*
(2) *$\hat{d}$ is quasiisometric to the word metric.*
(3) *The metric space $(G, \hat{d})$ is weakly geodesic and strongly bolic.*

This paper is organized as follows. In Sect. 2, we recall the concepts of hyperbolic groups and bicombings. In Sect. 3, we introduce a distance-like function r on a hyperbolic group and study its basic properties. In Sect. 4, we prove that r satisfies certain distance-like inequalities. In Sect. 5, we construct a metric $\hat{d}$ on a hyperbolic group and prove Theorem 17 stated above. In Sect. 6, we combine Lafforgue's work and Theorem 17 to prove the Baum-Connes conjecture for hyperbolic groups and their subgroups.

After this work was done, we learned from Vincent Lafforgue that he has independently proved the Baum-Connes conjecture for hyperbolic groups by a different and elegant method [14], and we also learned from Michael Puschnigg that he has independently proved the Kadison-Kaplansky conjecture for hyperbolic groups using a beautiful local cyclic homology method [17]. It is our pleasure to thank both of them for bringing their work to our attention.

We also would like to thank the referee for helpful suggestions.

2. Hyperbolic groups and bicombings

In this section, we recall the concepts of hyperbolic groups and bicombings.

2.1. Hyperbolic groups. Let G be a finitely generated group. Let S be a finite generating set for G. Recall that the Cayley graph of G with respect to S is the graph Γ satisfying the following conditions:

(1) the set of vertices in Γ, denoted by $\Gamma^{(0)}$, is G;
(2) the set of edges is $G \times S$, where each edge $(g, s) \in G \times S$ spans the vertices g and gs.

We endow Γ with the path metric d induced by assigning length 1 to each edge. Notice that G acts freely, isometrically and cocompactly on Γ. A geodesic path in Γ is a shortest edge path. The restriction of the path metric d to G is called the word metric.

A finitely generated group G is called hyperbolic if there exists a constant $\delta \geq 0$ such that all the geodesic triangles in Γ are δ-fine in the following sense: if a, b, and c are vertices in Γ, $[a, b]$, $[b, c]$, and $[c, a]$ are geodesics from a to b, from b to c, and from c to a, respectively, and points $\bar{a} \in [b, c]$, $v, \bar{c} \in [a, b]$, $w, \bar{b} \in [a, c]$ satisfy

$$d(b, \bar{c}) = d(b, \bar{a}), \quad d(c, \bar{a}) = d(c, \bar{b}),$$
$$d(a, v) = d(a, w) \leq d(a, \bar{c}) = d(a, \bar{b}),$$

then $d(v, w) \leq \delta$.

The above definition of hyperbolicity does not depend on the choice of the finite generating set S. See [6, 1] for other equivalent definitions.

For vertices a, b, and c in Γ, the Gromov product is defined by

$$(b|c)_a := d(a, \bar{b}) = d(a, \bar{c}) = \frac{1}{2}\Big[d(a, b) + d(a, c) - d(b, c)\Big].$$

The Gromov product can be used to measure the degree of cancellation in the multiplication of group elements in G.

2.2. Bicombings. Let G be a finitely generated group. Let Γ be a Cayley graph with respect to a finite generating set. A bicombing p in Γ is a function assigning to each ordered pair (a, b) of vertices in Γ an oriented edge-path $p[a, b]$ from a to b. A bicombing p is called geodesic if each path $p[a, b]$ is geodesic, i.e. a shortest edge path. A bicombing p is G-equivariant if $p[g \cdot a, g \cdot b] = g \cdot p[a, b]$ for each $a, b \in \Gamma^{(0)}$ and each $g \in G$.

3. Definition and properties of $r(a, b)$

The purpose of this section is to introduce a distance-like function r on a hyperbolic group and study its basic properties.

Let G be a hyperbolic group and Γ be a Cayley graph of G with respect to a finite generating set. We endow Γ with the path metric d, and identify G with $\Gamma^{(0)}$, the set of vertices of Γ. Let $\delta \geq 1$ be a positive integer such that all the geodesic triangles in Γ are δ-fine.

The ball $B(x, R)$ is the set of all vertices at distance at most R from the vertex x. The sphere $S(x, R)$ is the set of all vertices at distance R from the vertex x. Pick an equivariant geodesic bicombing p in Γ. By $p[a, b](t)$ we denote the point on the geodesic path $p[a, b]$ at distance t from a. Recall that $C_0(G, \mathbb{Q})$ is the space of all 0-chains (in $G = \Gamma^{(0)}$) with coefficients in $\mathbb{Q}$. Endow $C_0(G, \mathbb{Q})$ with the ℓ^1-norm $|\cdot|_1$. We identify G with the standard basis of $C_0(G, \mathbb{Q})$. Therefore the left action of G on itself induces a left action on $C_0(G, \mathbb{Q})$.

First we recall several constructions from [16].

For $v, w \in G$, the flower at w with respect to v is defined to be

$$Fl(v, w) := S(v, d(v, w)) \cap B(w, \delta) \subseteq G.$$

For each $a \in G$, we define $pr_a : G \to G$ by:

(1) $pr_a(a) := a$;
(2) if $b \neq a$, $pr_a(b) := p[a,b](t)$, where t is the largest integral multiple of 10δ which is strictly less than $d(a,b)$.

Now for each pair $a, b \in G$, we define a 0-chain $f(a,b)$ in G inductively on the distance $d(a,b)$ as follows:

(1) if $d(a,b) \leq 10\delta$, $f(a,b) := b$;
(2) if $d(a,b) > 10\delta$ and $d(a,b)$ is not an integral multiple of 10δ, let $f(a,b) := f(a, pr_a(b))$;
(3) if $d(a,b) > 10\delta$ and $d(a,b)$ is an integral multiple of 10δ, let

$$f(a,b) := \frac{1}{\#Fl(a,b)} \sum_{x \in Fl(a,b)} f(a, pr_a(x)).$$

Proposition 1 ([16]). *The function* $f : G \times G \to C_0(G, \mathbb{Q})$ *defined above satisfies the following conditions.*

(1) *For each* $a, b \in G$, $f(b,a)$ *is a* convex combination, *i.e. its coefficients are non-negative and sum up to 1.*
(2) *If* $d(a,b) \geq 10\delta$, *then* $\operatorname{supp} f(b,a) \subseteq B(p[b,a](10\delta), \delta) \cap S(b, 10\delta)$.
(3) *If* $d(a,b) \leq 10\delta$, *then* $f(b,a) = a$.
(4) f *is* G*-equivariant, i.e.* $f(g \cdot b, g \cdot a) = g \cdot f(b,a)$ *for any* $g, a, b \in G$.
(5) *There exist constants* $L \geq 0$ *and* $0 \leq \lambda < 1$ *such that, for all* $a, a', b \in G$,

$$\left| f(b,a) - f(b,a') \right|_1 \leq L\, \lambda^{(a|a')_b}.$$

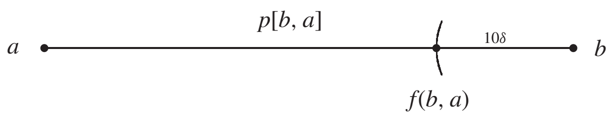

Fig. 3.1 Convex combination $f(b,a)$

Let ω_7 be the number of elements in a ball of radius 7δ in G. For each $a \in G$, a 0-chain $star(a)$ is defined by

$$star(a) := \frac{1}{\omega_7} \sum_{x \in B(a, 7\delta)} x.$$

This extends to a linear operator $star : C_0(G, \mathbb{Q}) \to C_0(G, \mathbb{Q})$. Define the 0-chain $\bar{f}(b,a)$ by $\bar{f}(b,a) := star\big(f(b,a)\big)$.

The main reason for introducing $\bar{f}$ is that $\bar{f}$ has better cancellation properties than f (compare Proposition 1(5) with Proposition 2(5) and 2(6) below). These cancellation properties play key roles in this paper.

Proposition 2 ([16]). *The function* $\bar{f} : G \times G \to C_0(G, \mathbb{Q})$ *defined above satisfies the following conditions.*

(1) *For each* $a, b \in G$, $\bar{f}(b, a)$ *is a convex combination.*
(2) *If* $d(a, b) \geq 10\delta$, *then* $\operatorname{supp} \bar{f}(b, a) \subseteq B(p[b, a](10\delta), 8\delta)$.
(3) *If* $d(a, b) \leq 10\delta$, *then* $\operatorname{supp} \bar{f}(b, a) \subseteq B(a, 7\delta)$.
(4) $\bar{f}$ *is G-equivariant, i.e.* $\bar{f}(g \cdot b, g \cdot a) = g \cdot \bar{f}(b, a)$ *for any* $g, a, b \in G$.
(5) *There exist constants* $L \geq 0$ *and* $0 \leq \lambda < 1$ *such that, for all* $a, a', b \in G$,
$$\left| \bar{f}(b, a) - \bar{f}(b, a') \right|_1 \leq L\, \lambda^{(a|a')_b}.$$
(6) *There exists a constant* $0 \leq \lambda' < 1$ *such that if* $a, b, b' \in G$ *satisfy* $(a|b)_{b'} \leq 10\delta$ *and* $(a|b')_b \leq 10\delta$, *then* $\left| \bar{f}(b, a) - \bar{f}(b', a) \right|_1 \leq 2\lambda'$.
(7) *Let* $a, b, c \in G$, γ *be a geodesic path from a to b, and let*
$$c \in N_G(\gamma, 9\delta) := \{x \in G \mid d(x, \gamma) \leq 9\delta\}.$$
Then $\operatorname{supp}(\bar{f}(c, a)) \subseteq N_G(\gamma, 9\delta)$.

Definition 3. For each pair of vertices $a, b \in G$, a rational number $r(a, b) \geq 0$ is defined inductively on $d(a, b)$ as follows.

- $r(a, a) := 0$.
- If $0 < d(a, b) \leq 10\delta$, let $r(a, b) := 1$.
- If $d(a, b) > 10\delta$, let $r(a, b) := r\big(a, \bar{f}(b, a)\big) + 1$, where $r\big(a, \bar{f}(b, a)\big)$ is defined by linearity in the second variable.

The function r is well defined by Proposition 2(2). Also, $r(a, b)$ is well defined when b is a 0-chain, by linearity.

Let $\mathbb{Q}_{\geq 0}$ denote the set of all non-negative rational numbers.

Proposition 4. *For the function* $r : G \times G \to \mathbb{Q}_{\geq 0}$ *defined above, there exists* $N \geq 0$ *such that, for all* $a, b, b' \in G$,
$$\left| r(a, b) - r(a, b') \right| \leq d(b, b') + N.$$

Proof. Up to the G-action, there are only finitely many triples of vertices a, b, b', satisfying $d(a, b) + d(a, b') \leq 40\delta$, hence there exists a uniform bound N' for the norms
$$\left| r(a, b) - r(a, b') \right|$$
for such vertices a, b, b'. Let λ' be the constant from Proposition 2(6) and pick N large enough so that

$$N' \leq N \qquad \text{and} \qquad \lambda' \cdot [27\delta + N] \leq N. \tag{3.1}$$

We shall prove the inequality in Proposition 4 by induction on $d(a, b) + d(a, b')$.

If $d(a, b) + d(a, b') \leq 40\delta$, then

$$\left|r(a, b) - r(a, b')\right| \leq N' \leq N \leq d(b, b') + N$$

just by the choices of N' and N. We assume now that $d(a, b)+d(a, b') > 40\delta$. Consider the following two cases.

Case 1. $(a|b')_b > 10\delta$ *or* $(a|b)_{b'} > 10\delta$.

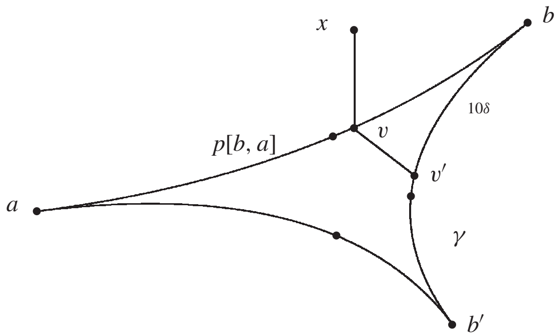

Fig. 3.2 Proposition 4, Case 1

Assume, for example, that $(a|b')_b > 10\delta$. Then $d(a, b) > 10\delta$, hence, by definition,

$$r(a, b) = r\bigl(a, \bar{f}(b, a)\bigr) + 1.$$

By Proposition 2(2), we have supp $\bar{f}(b, a) \subseteq B(v, 8\delta)$, where $v := p[b, a](10\delta)$. Also, $(a|b')_b > 10\delta$ implies $d(b, b') > 10\delta$. Hence there exists a geodesic γ between b and b', and a vertex v' on γ with $d(b, v') = d(b, v) = 10\delta$. Since geodesic triangles are δ-fine, $d(v, v') \leq \delta$. For every $x \in$ supp $\bar{f}(b, a)$,

$$\begin{aligned} d(x, b') &\leq d(x, v) + d(v, v') + d(v', b') \\ &\leq 8\delta + \delta + \bigl[d(b, b') - 10\delta\bigr] \\ &\leq d(b, b') - 1, \\ d(a, x) &\leq d(a, v) + d(v, x) \\ &\leq \bigl[d(a, b) - 10\delta\bigr] + 8\delta \\ &\leq d(a, b) - 1. \end{aligned}$$

Therefore

$$d(a, x) + d(a, b') < d(a, b) + d(a, b').$$

Hence the induction hypotheses apply to the vertices a, x, and b', giving

$$(3.2) \qquad \left|r(a,x) - r(a,b')\right| \le d(x,b') + N \le d(b,b') - 1 + N.$$

By Proposition 2(1–2),

$$\bar{f}(b,a) = \sum_{x\in B(v,8\delta)} \alpha_x x$$

for some non-negative coefficients α_x summing up to 1. By the definition of r and inequality (3.2), we have

$$\begin{aligned}
&\left|r(a,b) - r(a,b')\right| \\
&= \left|r\big(a, \bar{f}(b,a)\big) + 1 - r(a,b')\right| \\
&= \left|\sum_{x\in B(v,8\delta)} \alpha_x r(a,x) + 1 - r(a,b')\right| \\
&\le \left|\sum_{x\in B(v,8\delta)} \alpha_x \big[r(a,x) - r(a,b')\big]\right| + 1 \\
&\le \sum_{x\in B(v,8\delta)} \alpha_x \left|r(a,x) - r(a,b')\right| + 1 \\
&\le \sum_{x\in B(v,8\delta)} \alpha_x \big(d(b,b') - 1 + N\big) + 1 \\
&= d(b,b') + N.
\end{aligned}$$

Case 2. $(a|b')_b \le 10\delta$ *and* $(a|b)_{b'} \le 10\delta$.

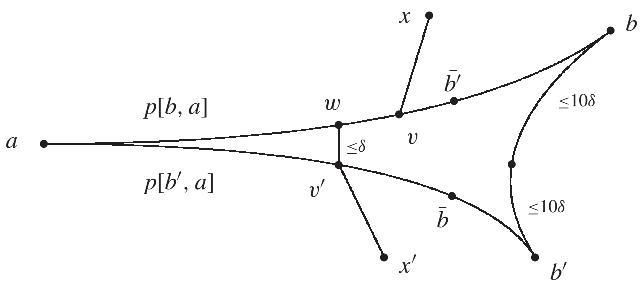

Fig. 3.3 Proposition 4, Case 2

Since $d(a,b) + d(a,b') > 40\delta$ and $d(b,b') = (a|b')_b + (a|b)_{b'} \le 20\delta$, we have $d(a,b) > 10\delta$ and $d(a,b') > 10\delta$. Then, by the definition of r,

(3.3)
$$\begin{aligned}&\left|r(a,b)-r(a,b')\right|\\&=\left|r\big(a,\bar f(b,a)\big)+1-r\big(a,\bar f(b',a)\big)-1\right|\\&=\left|r\big(a,\bar f(b,a)-\bar f(b',a)\big)\right|.\end{aligned}$$

The 0-chain $\bar f(b,a)-\bar f(b',a)$ can be represented in the form f_+-f_-, where f_+ and f_- are 0-chains with non-negative coefficients and disjoint supports. By Proposition 2(6),

$$\begin{aligned}|f_+|_1+|f_-|_1 &= |f_+-f_-|_1\\&=\left|\bar f(b,a)-\bar f(b',a)\right|_1\\&\le 2\lambda'.\end{aligned}$$

Since the coefficients of the 0-chain $f_+-f_-=\bar f(b,a)-\bar f(b',a)$ sum up to 0, then

(3.4)
$$|f_+|_1=|f_-|_1\le\lambda'.$$

With the notations $v:=p[b,a](10\delta)$, $v':=p[b',a](10\delta)$, we have

$$\begin{aligned}&\text{supp }f_+\subseteq\text{supp }\bar f(b,a)\subseteq B(v,8\delta)\quad\text{and}\\&\text{supp }f_-\subseteq\text{supp }\bar f(b',a)\subseteq B(v',8\delta)\end{aligned}$$

(see Fig. 3.3). Since geodesic triangles are δ-fine, there exists a point w on $p[b,a]$ such that $d(a,w)=d(a,v')$ and $d(w,v')\le\delta$. We first assume that $d(a,w)\le d(a,v)$. We have

$$\begin{aligned}d(v,v') &\le d(v,w)+d(w,v')\\&\le d(w,\overline{b'})+\delta\\&=d(v',\overline{b})+\delta\\&\le 11\delta,\end{aligned}$$

where $\overline{b'}$ and $\overline{b}$ are the inscribed points as in the definition of δ-fine triangle in Sect. 2.1. If $d(a,w)>d(a,v)$, we can apply the same argument to prove $d(v,v')\le 11\delta$ by interchanging v' with v.

Hence by Proposition 2(2), for each $x\in\text{supp }f_+$ and $x'\in\text{supp }f_-$,

$$\begin{aligned}d(x,x') &\le d(x,v)+d(v,v')+d(v',x')\\&\le 8\delta+11\delta+8\delta\\&=27\delta.\end{aligned}$$

Also $d(a,x)+d(a,x')<d(a,b)+d(a,b')$, so the induction hypotheses for the vertices a, x, and x' apply, giving

(3.5)
$$\begin{aligned}\left|r(a,x)-r(a,x')\right| &\le d(x,x')+N\\&\le 27\delta+N\end{aligned}$$

for each $x \in \operatorname{supp} f_+$ and $x' \in \operatorname{supp} f_-$. Then we continue equality (3.3) using (3.4), (3.5), linearity of r in the second variable, and the definition of N in (3.1):

$$\begin{aligned} \left|r(a,b) - r(a,b')\right| &= \left|r\big(a, \bar f(b,a) - \bar f(b',a)\big)\right| \\ &= \left|r(a, f_+) - r(a, f_-)\right| \\ &\le \lambda' \cdot \big[27\delta + N\big] \\ &\le N \le d(b,b') + N. \end{aligned}$$

Proposition 4 is proved. □

Let $\varepsilon : C_0(G, \mathbb{Q}) \to \mathbb{Q}$ be the augmentation map taking each 0-chain to the sum of its coefficients. A 0-chain z with $\varepsilon(z) = 0$ is called a 0-cycle.

Proposition 5. *There exists a constant $D \geq 0$ such that, for each $a \in G$ and each 0-cycle z,*

$$\left|r(a,z)\right| \le D\,|z|_1\, diam\big(\operatorname{supp}(z)\big).$$

Proof. It suffices to consider the case $z = b - b'$, where b and b' are vertices with $d(b,b') = 1$. But this case is immediate from Proposition 4 by taking $D := \frac{1}{2}(1+N)$. □

Theorem 6. *For a hyperbolic group G, the function $r : G \times G \to \mathbb{Q}_{\geq 0}$ from Definition 3 satisfies the following properties.*

(1) *r is G-equivariant, i.e. $r(a,b) = r(g \cdot a, g \cdot b)$ for $g, a, b \in G$.*
(2) *r is Lipschitz equivalent to the word metric. More precisely, we have*

$$\frac{1}{20\delta}\, d(a,b) \le r(a,b) \le d(a,b)$$

for all $a, b \in G$.

(3) *There exist constants $C \geq 0$ and $0 \leq \mu < 1$ such that, for all $a, a', b, b' \in G$ with $d(a,a') \leq 1$ and $d(b,b') \leq 1$,*

$$\left|r(a,b) - r(a',b) - r(a,b') + r(a',b')\right| \le C\mu^{d(a,b)}.$$

In particular, if $d(a,a') \leq 1$ and $d(b,b') \leq 1$, then

$$\left|r(a,b) - r(a',b) - r(a,b') + r(a',b')\right| \to 0 \quad as \quad d(a,b) \to \infty.$$

Proof. **(1)** The G-equivariance of r follows from the definition of r and Proposition 2(4).
(2) Using the assumption that $\delta \geq 1$ and the definition of r, the inequalities

$$\frac{1}{20\delta} d(a,b) \le r(a,b) \le d(a,b)$$

can be shown by an easy induction on $d(a,b)$. The remaining part **(3)** immediately follows from the following proposition. □

Proposition 7. *There exist constants* $A > 0$*,* $B > 0$*, and* $0 < \rho < 1$ *such that, for all* $a, a', b, b' \in G$ *with* $d(a, a') \leq 1$ *and* $d(b, b') \leq 30\delta$*,*

$$\left|r(a, b) - r(a', b) - r(a, b') + r(a', b')\right| \leq \big(A\, d(b, b') + B\big)\, \rho^{d(a,b)+d(a,b')}.$$

Proof. Let $D \geq 0$ be the constant from Proposition 5, $L \geq 0$ and $0 \leq \lambda < 1$ be the constants from Propositions 1(5) and 2(5), $\delta \geq 1$ be an integral hyperbolicity (fine-triangles) constant, and ω_7 be the number of vertices in a ball of radius 7δ in G.

Now we define constants A, B and ρ. Since the inequality obviously holds when $b = b'$, we will assume that $d(b, b') \geq 1$. Then constant $A > 0$ can be chosen large enough so that

- the desired inequality is satisfied whenever $d(a, b) + d(a, b') \leq 100\delta$, $\rho \geq \sqrt{\lambda}$, and $B > 0$, and
- $32D\delta L\big(\sqrt{\lambda}\big)^{-32\delta} < A$.

So from now on we can assume that $d(a, b) + d(a, b') > 100\delta$. Also the choice of A implies that inequalities

$$1 - \frac{A}{Al + B} + \frac{32D\delta L\big(\sqrt{\lambda}\big)^{t-32\delta}}{(Al + B)\rho^{t-18\delta}} \leq 1 - \frac{A}{Al + B} + \frac{32D\delta L\big(\sqrt{\lambda}\big)^{-32\delta}}{Al + B} < 1$$

hold for all $B > 0$, $\sqrt{\lambda} \leq \rho < 1$, $1 \leq l \leq 30\delta$, and $t \geq 0$. Therefore, we can pick $B > 0$ sufficiently large and $\rho < 1$ sufficiently close to 1 so that the inequalities

$$1 - \frac{A}{Al + B} + \frac{32D\delta L\big(\sqrt{\lambda}\big)^{t-32\delta}}{(Al + B)\rho^{t-18\delta}} \leq \rho^{18\delta} \qquad \text{and}$$

$$\left(1 - \frac{1}{\omega_7}\right) \frac{30\delta A + B}{B} + \frac{64D\delta L\big(\sqrt{\lambda}\big)^{t-32\delta}}{B\rho^{t-36\delta}} \leq \rho^{36\delta}$$

are satisfied for all $1 \leq l \leq 30\delta$ and all $t \geq 0$. The above inequalities rewrite as

(3.6)

$$\big(A(l - 1) + B\big)\, \rho^{t-18\delta} + 32D\delta L\big(\sqrt{\lambda}\big)^{t-32\delta} \leq \big(Al + B\big)\, \rho^{t} \qquad \text{and}$$

(3.7)

$$\left(1 - \frac{1}{\omega_7}\right) (30\delta A + B)\, \rho^{t-36\delta} + 64D\delta L\big(\sqrt{\lambda}\big)^{t-32\delta} \leq B\, \rho^{t}$$

and they are satisfied for all $1 \leq l \leq 30\delta$ and all $t \geq 0$.

The proof of the proposition proceeds by induction on $d(a, b) + d(a, b')$. We consider the following two cases.

Case 1. $(a|b)_{b'} > 10\delta$ *or* $(a|b')_b > 10\delta$.

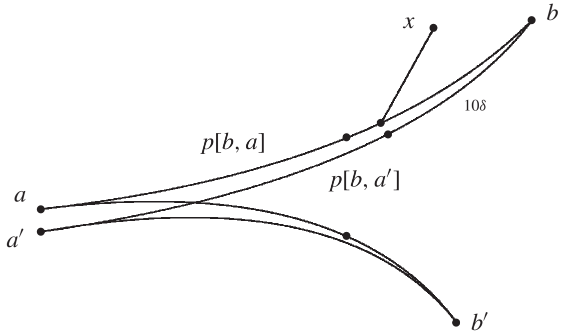

Fig. 3.4 Proposition 7, Case 1

Without loss of generality, $(a|b')_b > 10\delta$ (interchange b and b' otherwise). The 0-cycle $f(b,a) - f(b,a')$ can be uniquely represented as $f_+ - f_-$, where f_+ and f_- are 0-chains with non-negative coefficients, disjoint supports, and of the same ℓ^1-norm. We have

$$f(b,a) = f_0 + f_+ \quad \text{and} \quad f(b,a') = f_0 + f_-$$

for some 0-chain f_0 with non-negative coefficients (actually $f_0 = \min\{f(b,a),\, f(b,a')\}$). Denote $\alpha := |f_+|_1 = |f_-|_1 = \varepsilon(f_+) = \varepsilon(f_-)$, where ε is the augmentation map. Since $d(a,a') \le 1$, then

$$\begin{aligned}(a|a')_b &\ge \frac{1}{2}\left[d(a,b) + d(a',b) - 1\right]\\ &\ge \frac{1}{2}\left[d(a,b) + d(a,b') - 32\delta\right],\end{aligned}$$

and by Proposition 1(5),

$$\begin{aligned}\alpha &= \frac{1}{2}\left|f(b,a) - f(b,a')\right|_1 \\ &\le \frac{1}{2}L\lambda^{(a|a')_b} \\ &\le \frac{1}{2}L\left(\sqrt{\lambda}\right)^{d(a,b)+d(a,b')-32\delta}.\end{aligned} \tag{3.8}$$

By the definition of hyperbolicity in Sect. 2.1 and the assumptions $d(a,b) + d(a,b') > 100\delta$ and $d(b,b') \le 30\delta$, we have

$$d\big(p[b,a](10\delta),\, p[b,a'](10\delta)\big) \le \delta.$$

Hence there exists a vertex $x_0 \in B\big(p[b,a](10\delta), 8\delta\big) \cap B\big(p[b,a'](10\delta), 8\delta\big)$. By the definitions of r and $\bar{f}$,

$$\begin{aligned}
&\big|r(a,b) - r(a',b) - r(a,b') + r(a',b')\big| \\
&= \Big|r\big(a, \bar{f}(b,a)\big) + 1 - r\big(a', \bar{f}(b,a')\big) - 1 - r(a,b') + r(a',b')\Big| \\
&= \Big|r\big(a, star(f_0 + f_+)\big) - r\big(a', star(f_0 + f_-)\big) - r(a,b') + r(a',b')\Big| \\
&\le \Big|r\big(a, star(f_0) + \alpha x_0\big) - r\big(a', star(f_0) + \alpha x_0\big) - r(a,b') + r(a',b')\Big| + \\
&+ \Big|r\big(a, star(f_+) - \alpha x_0\big)\Big| + \Big|r\big(a', \alpha x_0 - star(f_-)\big)\Big|.
\end{aligned}$$

Now we bound each of the three terms in the last sum. We number these terms consecutively as T_1, T_2, T_3.

Term T_1. Using the same argument as in Case 1 in the proof of Proposition 4, one checks that, for each

$$x \in \operatorname{supp}\big(star(f_0) + \alpha x_0\big) \subseteq B(p[b,a](10\delta), 8\delta) \cap B(p[b,a'](10\delta), 8\delta),$$

the following conditions hold:

$$\begin{aligned}
&d(x,b') \le d(b,b') - 1 \le 30\delta \qquad \text{and} \\
&d(a,b) + d(a,b') - 18\delta \le d(a,x) + d(a,b') \le d(a,b) + d(a,b') - 1.
\end{aligned}$$

In particular, the induction hypotheses are satisfied for the vertices a, a', x, b', giving

$$\begin{aligned}
&\big|r(a,x) - r(a',x) - r(a,b') + r(a',b')\big| \\
&\le \big(A\, d(x,b') + B\big)\, \rho^{d(a,x)+d(a,b')} \\
&\le \Big(A\big(d(b,b') - 1\big) + B\Big)\, \rho^{d(a,b)+d(a,b')-18\delta}.
\end{aligned}$$

Since $star(f_0) + \alpha x_0$ is a convex combination, by linearity of r in the second variable,

$$\begin{aligned}
T_1 = &\ \Big|r\big(a, star(f_0) + \alpha x_0\big) - r\big(a', star(f_0) + \alpha x_0\big) - r(a,b') + r(a',b')\Big| \\
\le &\ \Big(A\,\big(d(b,b') - 1\big) + B\Big)\, \rho^{d(a,b)+d(a,b')-18\delta}.
\end{aligned}$$

Terms T_2 and T_3. Since $star(f_+) - \alpha x_0$ is a 0-cycle supported in a ball of radius 8δ, by Proposition 5 and inequality (3.8),

$$\begin{aligned}
T_2 = &\Big|r\big(a, star(f_+) - \alpha x_0\big)\Big| \\
\le &D\, \Big|star(f_+) - \alpha x_0\Big|_1 \cdot 16\delta \\
\le &D \cdot 2\alpha \cdot 16\delta \\
\le &16 D \delta L \big(\sqrt{\lambda}\big)^{d(a,b)+d(a,b')-32\delta}.
\end{aligned}$$

Analogously,

$$T_3 = \left| r\left(a', \alpha x_0 - star(f_-)\right) \right| \le 16D\delta L\left(\sqrt{\lambda}\right)^{d(a,b)+d(a,b')-32\delta}.$$

Combining the three bounds above and using the definition of B and ρ (inequality (3.6)),

$$\begin{aligned}
&\left| r(a,b) - r(a',b) - r(a,b') + r(a',b') \right| \\
&\le T_1 + T_2 + T_3 \\
&\le \Big(A\big(d(b,b')-1\big) + B\Big)\rho^{d(a,b)+d(a,b')-18\delta} + 32D\delta L\left(\sqrt{\lambda}\right)^{d(a,b)+d(a,b')-32\delta} \\
&\le \big(A\, d(b,b') + B\big)\, \rho^{d(a,b)+d(a,b')}.
\end{aligned}$$

This finishes Case 1.

Case 2. $(a|b)_{b'} \le 10\delta$ *and* $(a|b')_b \le 10\delta.$

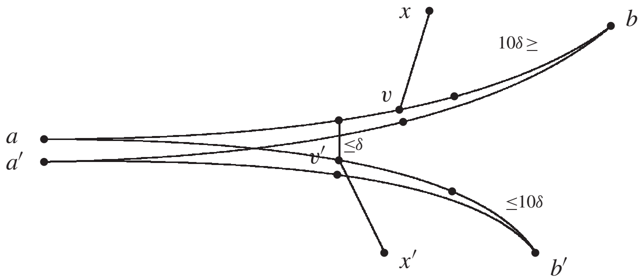

Fig. 3.5 Proposition 7, Case 2

As in Case 1, we have

$$\begin{gathered}
f(b,a) - f(b,a') = f_+ - f_-, \\
f(b,a) = f_0 + f_+, \quad f(b,a') = f_0 + f_-, \\
\alpha := |f_+|_1 = |f_-|_1 = \varepsilon(f_+) = \varepsilon(f_-), \\
\alpha \le \frac{1}{2} L\lambda^{(a|a')_b} \le \frac{1}{2} L\left(\sqrt{\lambda}\right)^{d(a,b)+d(a,b')-32\delta},
\end{gathered}$$

where f_+, f_-, and f_0 are 0-chains with non-negative coefficients, and f_+ and f_- have disjoint supports. Analogously, interchanging b and b',

$$\begin{gathered}
f(b',a) - f(b',a') = f'_+ - f'_-, \\
f(b',a) = f'_0 + f'_+, \quad f(b',a') = f'_0 + f'_-, \\
\alpha' := |f'_+|_1 = |f'_-|_1 = \varepsilon(f'_+) = \varepsilon(f'_-), \\
\alpha' \le \frac{1}{2} L\lambda^{(a|a')_{b'}} \le \frac{1}{2} L\left(\sqrt{\lambda}\right)^{d(a,b)+d(a,b')-32\delta},
\end{gathered}$$

where f'_+, f'_-, and f'_0 are 0-chains with non-negative coefficients, and f'_+ and f'_- have disjoint supports.

Denote $v := p[b, a](10\delta)$ and $v' := p[b', a](10\delta)$. By the conditions of Case 2 and δ-hyperbolicity of Γ, using the same argument as in Case 2 in the proof of Proposition 4, we obtain $d(v, v') \le 11\delta$. Let x_0 be a vertex closest to the mid-point of a geodesic path connecting v to v'. Proposition 1(2) implies that

$$\begin{aligned}&\operatorname{supp}\ f_0 \cup \operatorname{supp}\ f'_0 \subseteq B(x_0, 7\delta) \qquad \text{and}\\ &\operatorname{supp}\ f_- \cup \operatorname{supp}\ f_+ \cup \operatorname{supp}\ f'_- \cup \operatorname{supp}\ f'_+ \subseteq B(x_0, 8\delta).\end{aligned}$$

By the definition of r,

$$\begin{aligned}&\left|r(a, b) - r(a', b) - r(a, b') + r(a', b')\right|\\ &= \left|r\big(a, \bar f(b, a)\big) - r\big(a', \bar f(b, a')\big) - r\big(a, \bar f(b', a)\big) + r\big(a', \bar f(b', a')\big)\right|\\ &\le \Big|r\big(a, star(f_0 + f_+)\big) - r\big(a', star(f_0 + f_-)\big) -\\ &\quad - r\big(a, star(f'_0 + f'_+)\big) + r\big(a', star(f'_0 + f'_-)\big)\Big|\\ &\le \Big|r\big(a, star(f_0) + \alpha x_0 - star(f'_0) - \alpha' x_0\big) -\\ &\quad - r\big(a', star(f_0) + \alpha x_0 - star(f'_0) - \alpha' x_0\big)\Big| +\\ &\quad + \left|r\big(a, star(f_+)\big) - r\big(a, \alpha x_0\big)\right| + \left|r\big(a', \alpha x_0\big) - r\big(a', star(f_-)\big)\right| +\\ &\quad + \left|r\big(a, \alpha' x_0\big) - r\big(a, star(f'_+)\big)\right| + \left|r\big(a', star(f'_-)\big) - r\big(a', \alpha' x_0\big)\right|.\end{aligned}$$

Now we bound each of the five terms in the last sum. We number these terms consecutively as S_1, ..., S_5.

<u>Term S_1</u>. One checks that, for each

$$\begin{aligned}&x \in \operatorname{supp}\big(star(f_0) + \alpha x_0\big) \subseteq B(v, 8\delta) \cap B(p[b, a'](10\delta), 8\delta) \quad \text{and}\\ &x' \in \operatorname{supp}\big(star(f'_0) + \alpha' x_0\big) \subseteq B(v', 8\delta) \cap B(p[b', a'](10\delta), 8\delta),\end{aligned}$$

the following conditions hold:

$$\begin{aligned}&d(x, x') \le 30\delta \quad \text{and}\\ &d(a, b) + d(a, b') - 36\delta \le d(a, x) + d(a, x') \le d(a, b) + d(a, b') - 1.\end{aligned}$$

In particular, the induction hypotheses are satisfied for the vertices a, a', x, x', giving

$$\begin{aligned}&\left|r(a, x) - r(a', x) - r(a, x') + r(a', x')\right|\\ &\le (Ad(x, x') + B)\, \rho^{d(a,x)+d(a,x')}\\ &\le (30\delta A + B)\, \rho^{d(a,b)+d(a,b')-36\delta}.\end{aligned} \tag{3.9}$$

Recall that ω_7 is the number of vertices in a ball of radius 7δ. Let β be the (positive) coefficient of x_0 in the 0-chain $star(f_0)$, and β' be the (positive) coefficient of x_0 in the 0-chain $star(f_0')$. Without loss of generality, we can assume $\left|star(f_0)\right|_1 \le \left|star(f_0')\right|_1$. Since x_0 was chosen so that supp $f_0 \cup$ supp $f_0' \subseteq B(x_0, 7\delta)$, by the definition of $star$, we have

$$\beta = \frac{1}{\omega_7}\left|f_0\right|_1 = \frac{1}{\omega_7}\left|star(f_0)\right|_1 \le \frac{1}{\omega_7}\left|star(f_0')\right|_1 = \frac{1}{\omega_7}\left|f_0'\right|_1 = \beta' \quad \text{and}$$

$$\alpha - \alpha' = \left(1 - \left|f_0\right|_1\right) - \left(1 - \left|f_0'\right|_1\right) = \left|f_0'\right|_1 - \left|f_0\right|_1 = \omega_7(\beta' - \beta) \ge 0.$$

Therefore,

$$\begin{aligned}
&\left|star(f_0) + \alpha x_0 - star(f_0') - \alpha' x_0\right|_1 \\
&\le \left|star(f_0) - \beta x_0\right|_1 + \left|\beta' x_0 - star(f_0')\right|_1 + \left|\left[(\alpha - \alpha') - (\beta' - \beta)\right] x_0\right|_1 \\
&= \left(\left|star(f_0)\right|_1 - \beta\right) + \left(\left|star(f_0')\right|_1 - \beta'\right) + \left(\beta' - \beta\right)\left(\omega_7 - 1\right) \\
&= \left(\left|f_0\right|_1 - \beta\right) + \left(\left|f_0'\right|_1 - \beta'\right) + \left(\left|f_0'\right|_1 - \left|f_0\right|_1\right)\left(1 - \frac{1}{\omega_7}\right) \\
&= \left|f_0\right|_1\left(1 - \frac{1}{\omega_7}\right) + \left|f_0'\right|_1\left(1 - \frac{1}{\omega_7}\right) + \left(\left|f_0'\right|_1 - \left|f_0\right|_1\right)\left(1 - \frac{1}{\omega_7}\right) \\
&= 2\left|f_0'\right|_1\left(1 - \frac{1}{\omega_7}\right) \\
&\le 2\left(1 - \frac{1}{\omega_7}\right).
\end{aligned}$$

Since $\left[star(f_0) + \alpha x_0\right] - \left[star(f_0') + \alpha' x_0\right]$ is a 0-cycle, it is of the form $h_+ - h_-$, where h_+ and h_- are 0-chains with non-negative coefficients, disjoint supports and of the same ℓ^1-norm, so we can define

$$\gamma := |h_+|_1 = |h_-|_1 = \varepsilon(h_+) = \varepsilon(h_-).$$

By the above inequality,

$$\gamma = \frac{1}{2}\,|h_+ - h_-|_1 \le 1 - \frac{1}{\omega_7},$$

then, by (3.9) and linearity of r in the second variable,

$$\begin{aligned}
S_1 &= \left|r(a, h_+ - h_-) - r(a', h_+ - h_-)\right| \\
&= \left|r(a, h_+) - r(a', h_+) - r(a, h_-) + r(a', h_-)\right| \\
&\le \gamma \cdot \left(30\delta A + B\right)\rho^{d(a,b)+d(a,b')-36\delta} \\
&\le \left(1 - \frac{1}{\omega_7}\right)\left(30\delta A + B\right)\rho^{d(a,b)+d(a,b')-36\delta}.
\end{aligned}$$

Terms $S_2 - S_5$. Analogously to term T_2 in Case 1,

$$\begin{aligned}
S_2 &\le 16D\delta L(\sqrt{\lambda})^{d(a,b)+d(a,b')-32\delta},\\
S_3 &\le 16D\delta L(\sqrt{\lambda})^{d(a,b)+d(a,b')-32\delta},\\
S_4 &\le 16D\delta L(\sqrt{\lambda})^{d(a,b)+d(a,b')-32\delta},\\
S_5 &\le 16D\delta L(\sqrt{\lambda})^{d(a,b)+d(a,b')-32\delta}.
\end{aligned}$$

Combining the bounds for the five terms above and using the definition of B and ρ (inequality (3.7)),

$$\begin{aligned}
&\left|r(a, b) - r(a', b) - r(a, b') + r(a', b')\right|\\
&\le S_1 + S_2 + S_3 + S_4 + S_5\\
&\le \left(1 - \frac{1}{\omega_7}\right)(30\delta A + B)\,\rho^{d(a,b)+d(a,b')-36\delta} + 64D\delta L(\sqrt{\lambda})^{d(a,b)+d(a,b')-32\delta}\\
&\le B\,\rho^{d(a,b)+d(a,b')}\\
&\le \left(A\,d(b, b') + B\right)\rho^{d(a,b)+d(a,b')}.
\end{aligned}$$

Proposition 7 and Theorem 6 are proved. □

4. More properties of r

In this section, we prove two distance-like inequalities for the function r introduced in the previous section.

As before, let G be a hyperbolic group and Γ be the Cayley graph of G with respect to a finite generating set. For any subset $A \subseteq \Gamma$, denote

$$N_G(A, R) := \{x \in G \mid d(x, A) \le R\}.$$

Proposition 8. *There exists $C_1 \ge 0$ with the following property. If $a, b \in G$, γ is a geodesic in Γ connecting a and b, $x \in G \cap \gamma$, γ' is the part of γ between x and b, and $c \in N_G(\gamma', 9\delta)$, then*

$$\left|r(a, c) - r(a, x) - r(x, c)\right| \le C_1 \qquad (\textit{Fig}.\ 4.1).$$

Proof. Let

$$C_1 := (80\delta + N + 36\delta DL)\sum_{k=0}^{\infty}\lambda^{k-18\delta},$$

where $L \ge 1$ and $0 < \lambda < 1$ are as in Propositions 1(5) and 2(5), N is as in Proposition 4, and D is as in Proposition 5. It suffices to show the inequality

$$\left|r(a, c) - r(a, x) - r(x, c)\right| \le (80\delta + N + 36\delta DL)\sum_{k=0}^{d(x,c)}\lambda^{k-18\delta}.$$

We will prove it by induction on $d(x, c)$.

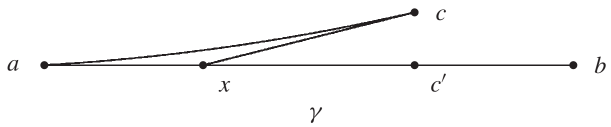

Fig. 4.1 Proposition 8

If $d(x, c) \leq 40\delta$, by Proposition 4 and Theorem 6(2) we have

$$\begin{aligned}
&\big|r(a, c) - r(a, x) - r(x, c)\big| \leq \big|r(a, c) - r(a, x)\big| + r(x, c) \\
&\leq \big(d(c, x) + N\big) + d(x, c) \leq 80\delta + N \\
&\leq (80\delta + N + 36\delta DL) \sum_{k=0}^{d(x,c)} \lambda^{k-18\delta}.
\end{aligned}$$

Now we assume that $d(x, c) > 40\delta$. There exists a vertex $c' \in \gamma'$ with $d(c', c) \leq 9\delta$, so

$$d(a, c) \geq d(a, c') - 9\delta \geq d(x, c') - 9\delta \geq d(x, c) - 18\delta > 10\delta.$$

Hence by the definition of the function r, we have

$$r(a, c) = r\big(a, \bar{f}(c, a)\big) + 1 \qquad \text{and} \qquad r(x, c) = r\big(x, \bar{f}(c, x)\big) + 1.$$

Also

$$\begin{aligned}
(a|x)_c &= \frac{1}{2}\big[d(c, a) + d(c, x) - d(a, x)\big] \\
&\geq \frac{1}{2}\big[d(c', a) - 9\delta + d(c', x) - 9\delta - d(a, x)\big] \\
&= d(x, c') - 9\delta \\
&\geq d(x, c) - 18\delta.
\end{aligned}$$

By Proposition 2(5),

$$\Big|\bar{f}(c, x) - \bar{f}(c, a)\Big|_1 \leq L\lambda^{(a|x)_c} \leq L\lambda^{d(x,c)-18\delta}.$$

This, together with Proposition 5 and Proposition 2(2), implies that

$$\begin{aligned}
\Big|r\big(a, \bar{f}(c, a)\big) - r\big(a, \bar{f}(c, x)\big)\Big| &= \Big|r\big(a, \bar{f}(c, a) - \bar{f}(c, x)\big)\Big| \\
&\leq DL\lambda^{d(x,c)-18\delta}\, diam\big(\operatorname{supp}(\bar{f}(c, a) - \bar{f}(c, x))\big) \\
&\leq 36\delta DL\lambda^{d(x,c)-18\delta}.
\end{aligned}$$

By Proposition 2(2) and 2(7), for every $y \in \operatorname{supp}(\bar{f}(c, x))$, we have

$$d(x, y) \leq d(x, c) - 1 \qquad \text{and} \qquad y \in N_G(\gamma', 9\delta).$$

Hence by the induction hypotheses, we obtain

$$\begin{aligned}
&\left|r(a, c) - r(a, x) - r(x, c)\right| \\
&= \left|\big(r\big(a, \bar{f}(c, a)\big) + 1\big) - r(a, x) - \big(r(x, \bar{f}(c, x)) + 1\big)\right| \\
&\leq \left|r\big(a, \bar{f}(c, a)\big) - r\big(a, \bar{f}(c, x)\big)\right| \\
&\quad + \left|r\big(a, \bar{f}(c, x)\big) - r(a, x) - r\big(x, \bar{f}(c, x)\big)\right| \\
&\leq 36\delta DL\lambda^{d(x,c)-18\delta} + (80\delta + N + 36\delta DL) \sum_{k=0}^{d(x,c)-1} \lambda^{k-18\delta} \\
&\leq (80\delta + N + 36\delta DL) \sum_{k=0}^{d(x,c)} \lambda^{k-18\delta}.
\end{aligned}$$

□

Proposition 9. *There exists $M' \geq 0$ such that*

$$\left|r(a, b) - r(a', b)\right| \leq M' \, d(a, a')$$

for all $a, a', b \in G$.

Proof. Recall that $\delta \geq 1$. Let

$$M' := (20\delta + 3 + 36\delta DL) \sum_{k=0}^{\infty} \lambda^{k-19\delta}.$$

The Cayley graph Γ is a geodesic metric space, hence it suffices to show the inequality

$$\left|r(a, b) - r(a', b)\right| \leq (20\delta + 3 + 36\delta DL) \sum_{k=0}^{d(b,a)} \lambda^{k-19\delta}$$

when $d(a, a') = 1$. We will prove it by induction on $d(a, b)$.

If $d(a, b) \leq 10\delta + 1$, then by Theorem 6(2) we have

$$\begin{aligned}
\left|r(a, b) - r(a', b)\right| &\leq r(a, b) + r(a', b) \\
&\leq d(a, b) + d(a', b) \\
&\leq 20\delta + 3 \\
&\leq (20\delta + 3 + 36\delta DL) \sum_{k=0}^{d(b,a)} \lambda^{k-19\delta}.
\end{aligned}$$

If $d(a, b) > 10\delta + 1$, then $d(a', b) > 10\delta$.

For every $y \in \operatorname{supp}(\bar{f}(b, a)) \cup \operatorname{supp}(\bar{f}(b, a'))$, by Proposition 2(2) we have

$$(a|a')_y = \frac{1}{2}\big[d(y, a) + d(y, a') - d(a, a')\big] \geq d(b, a) - 19\delta.$$

Hence by the definition of the function r, the induction hypothesis and Propositions 2(5) and 5, we obtain

$$\begin{aligned}
&\big|r(a, b) - r(a', b)\big| \\
&= \Big|\big(r(a, \bar{f}(b, a)) + 1\big) - \big(r(a', \bar{f}(b, a')) + 1\big)\Big| \\
&\leq \big|r(a, \bar{f}(b, a)) - r(a', \bar{f}(b, a))\big| + \big|r(a', \bar{f}(b, a)) - r(a', \bar{f}(b, a'))\big| \\
&\leq (20\delta + 3 + 36\delta DL) \sum_{k=0}^{d(b,a)-1} \lambda^{k-19\delta} \\
&\quad + DL\lambda^{d(b,a)-19\delta} diam\Big(\operatorname{supp}\big(\bar{f}(b, a) - \bar{f}(b, a')\big)\Big) \\
&\leq (20\delta + 3 + 36\delta DL) \sum_{k=0}^{d(b,a)-1} \lambda^{k-19\delta} + 36\delta DL\lambda^{d(b,a)-19\delta} \\
&\leq (20\delta + 3 + 36\delta DL) \sum_{k=0}^{d(b,a)} \lambda^{k-19\delta}.
\end{aligned}$$

□

5. Definition and properties of a new metric $\hat{d}$

In this section, we use the function r defined in Sect. 3 to construct a G-invariant metric $\hat{d}$ on a hyperbolic group G such that $\hat{d}$ is quasi-isometric to the word metric and prove that $(G, \hat{d})$ is weakly geodesic and strongly bolic.

We define

$$s(a, b) := \frac{1}{2}\big[r(a, b) + r(b, a)\big]$$

for all $a, b \in G$.

Proposition 10. *The above function s satisfies the following conditions.*

(a) *There exists $M \geq 0$ such that*

$$\big|s(u, v) - s(u, v')\big| \leq M\, d(v, v') \quad \textit{and} \quad \big|s(u, v) - s(u', v)\big| \leq M\, d(u, u')$$

for all $u, u', v, v' \in G$.

(b) *There exists* $C_1 \geq 0$ *such that if a vertex* w *lies on a geodesic connecting vertices* u *and* v, *then*

$$\left| s(u,v) - s(u,w) - s(w,v) \right| \leq C_1.$$

Proof. **(a)** Since s is symmetric, it suffices to show only the first inequality. Since the Cayley graph Γ is a geodesic metric space, it suffices to consider only the case $d(v,v') = 1$. This case follows from Propositions 4 and 9.

(b) follows from Proposition 8. □

Proposition 11. *There exists* $C_2 \geq 0$ *such that*

$$s(a,b) \leq s(a,c) + s(c,b) + C_2$$

for all $a, b, c \in G$.

Proof. Let $\bar{a} \in p[b,c]$, $\bar{c} \in p[a,b]$, $\bar{b} \in p[a,c]$ such that

$$d(b,\bar{c}) = d(b,\bar{a}), \quad d(c,\bar{a}) = d(c,\bar{b}), \quad d(a,\bar{c}) = d(a,\bar{b}).$$

By the definition of hyperbolicity, we have

$$d(\bar{a},\bar{b}) \leq \delta, \quad d(\bar{a},\bar{c}) \leq \delta, \quad d(\bar{b},\bar{c}) \leq \delta.$$

By Proposition 10,

$$\begin{aligned} s(a,b) &\leq s(a,\bar{c}) + s(\bar{c},b) + C_1 \\ &\leq \big(s(a,\bar{b}) + M\, d(\bar{b},\bar{c})\big) + \big(s(\bar{a},b) + M\, d(\bar{c},\bar{a})\big) + C_1 \\ &\leq s(a,\bar{b}) + s(\bar{a},b) + 2\delta M + C_1 \\ &\leq \big(s(a,\bar{b}) + s(\bar{b},c)\big) + \big(s(c,\bar{a}) + s(\bar{a},b)\big) + 2\delta M + C_1 \\ &\leq s(a,c) + s(c,b) + 2\delta M + 3C_1, \end{aligned}$$

so we set $C_2 := 2\delta M + 3C_1$. □

For every pair of elements $a, b \in G$, we define

$$\hat{d}(a,b) := \begin{cases} s(a,b) + C_2 & \text{if } a \neq b, \\ 0 & \text{if } a = b. \end{cases}$$

Proposition 12. *The function* $\hat{d}$ *defined above is a metric on* G.

Proof. By definition, $\hat{d}$ is symmetric, and $\hat{d}(a,b) = 0$ iff $a = b$. The triangle inequality is a direct consequence of Proposition 11. □

Proposition 13. *There exist constants $C \geq 0$ and $0 \leq \mu < 1$ with the following property. For all $R \geq 0$ and all $a, a', b, b' \in G$ with $d(a, a') \leq R$ and $d(b, b') \leq R$,*

$$\left|\hat{d}(a, b) - \hat{d}(a', b) - \hat{d}(a, b') + \hat{d}(a', b')\right| \leq R^2 C \mu^{d(a,b)-2R}.$$

In particular, if $d(a, a') \leq R$ and $d(b, b') \leq R$, then

$$\hat{d}(a, b) - \hat{d}(a', b) - \hat{d}(a, b') + \hat{d}(a', b') \to 0 \quad \text{as} \quad d(a, b) \to \infty.$$

Proof. Take C and μ as in Theorem 6(3). Increasing C if needed we can assume that $a \neq b, a \neq b', a' \neq b, a' \neq b'$.

If $a = a'$ or $b = b'$, then

$$\hat{d}(a, b) - \hat{d}(a', b) - \hat{d}(a, b') + \hat{d}(a', b') = 0.$$

If $d(a, a') = 1$ and $d(b, b') = 1$, then by Theorem 6(3),

$$\begin{aligned}
&\left|\hat{d}(a, b) - \hat{d}(a', b) - \hat{d}(a, b') + \hat{d}(a', b')\right| \\
&= \left|s(a, b) - s(a', b) - s(a, b') + s(a', b')\right| \\
&\leq C\mu^{d(a,b)}.
\end{aligned}$$

Without loss of generality, we can assume that R is an integer. In the general case

$$d(a, a') \leq R \qquad \text{and} \qquad d(b, b') \leq R,$$

pick vertices $a = a_0, a_1, ..., a_R = a'$ with $d(a_{i-1}, a_i) \leq 1$ and $b = b_0, b_1, ..., b_R = b'$ with $d(b_{j-1}, b_j) \leq 1$ and note that $d(a_i, b_j) \geq d(a, b) -2R$. Then we have

$$\begin{aligned}
&\left|\hat{d}(a, b) - \hat{d}(a', b) - \hat{d}(a, b') + \hat{d}(a', b')\right| \\
&= \left|s(a, b) - s(a', b) - s(a, b') + s(a', b')\right| \\
&= \Big|\sum_{i=1}^{R}\sum_{j=1}^{R}\big(s(a_{i-1}, b_{j-1}) - s(a_i, b_{j-1}) - s(a_{i-1}, b_j) + s(a_i, b_j)\big)\Big| \\
&\leq \sum_{i=1}^{R}\sum_{j=1}^{R}\left|s(a_{i-1}, b_{j-1}) - s(a_i, b_{j-1}) - s(a_{i-1}, b_j) + s(a_i, b_j)\right| \\
&\leq R^2 C\mu^{d(a,b)-2R}.
\end{aligned}$$

□

Recall that a metric space (X, d) is said to be weakly geodesic [11,10] if there exists $\delta_1 \geq 0$ such that, for every pair of points x and y in X and every $t \in [0, d(x, y)]$, there exists a point $a \in X$ such that $d(a, x) \leq t + \delta_1$ and $d(a, y) \leq d(x, y) - t + \delta_1$.

Proposition 14. *The metric space $(G, \hat{d})$ is weakly geodesic.*

Proof. Let $x, y \in G$ and $z \in G \cap p[x, y]$. By the definition of $\hat{d}$ and Proposition 10(b), we have

$$\hat{d}(x, z) + \hat{d}(z, y) - \hat{d}(x, y) \le C_1 + 2C_2.$$

It follows that

$$\hat{d}(x, z) \le \hat{d}(x, y) + C_1 + 2C_2,$$

hence the image of the map

$$\hat{d}(x, \cdot) : G \cap p[x, y] \to [0, \infty),$$

is contained in $[0, \hat{d}(x, y) + C_1 + 2C_2]$. Also, the image contains 0 and $\hat{d}(x, y)$.

By Proposition 10(a), we have

$$\left|\hat{d}(x, z') - \hat{d}(x, z)\right| \le M$$

when $d(z', z) = 1$. This, together with the fact that $p[x, y]$ is a geodesic path, implies that the image of the map

$$\hat{d}(x, \cdot) : G \cap p[x, y] \to [0, \hat{d}(x, y) + C_1 + 2C_2]$$

is M-dense in $[0, \hat{d}(x, y)]$, i.e. for every $t \in [0, \hat{d}(x, y)]$, there exists $a \in G \cap p[x, y]$ such that

$$\left|\hat{d}(x, a) - t\right| \le M.$$

It follows that $\hat{d}(x, a) \le t + M$, and by Proposition 10(b) we also have

$$\left|\hat{d}(x, y) - \hat{d}(x, a) - \hat{d}(a, y)\right| \le C_1 + 2C_2.$$

This implies that

$$\begin{aligned} \hat{d}(a, y) &\le \hat{d}(x, y) - \hat{d}(x, a) + C_1 + 2C_2 \\ &\le \hat{d}(x, y) - t + M + C_1 + 2C_2. \end{aligned}$$

Therefore $(G, \hat{d})$ is weakly geodesic for $\delta_1 := M + C_1 + 2C_2$. □

Kasparov and Skandalis introduced the concept of bolicity in [11, 10].

Definition 15. A metric space (X, d) is said to be bolic if there exists $\delta_2 \ge 0$ with the following properties:

(B1) for any $R > 0$, there exists $R' > 0$ such that for all $a, a', b, b' \in X$ satisfying

$$d(a, a') + d(b, b') \le R \qquad \text{and} \qquad d(a, b) + d(a', b') \ge R',$$

we have

$$d(a, b') + d(a', b) \le d(a, b) + d(a', b') + 2\delta_2; \qquad \text{and}$$

(B2) there exists a map $m : X \times X \to X$, such that, for all $x, y, z \in X$, we have

$$2d(m(x, y), z) \leq \left(2d(x, z)^2 + 2d(y, z)^2 - d(x, y)^2\right)^{\frac{1}{2}} + 4\delta_2.$$

(X, d) is called strongly bolic if it is bolic and the above condition (B1) holds for every $\delta_2 > 0$ [13].

Proposition 16. *The metric space $(G, \hat{d})$ is strongly bolic.*

Proof. Proposition 13 yields condition (B1) for all $\delta_2 > 0$. It remains to show that there exist $\delta_2 \geq 0$ and a map $m : G \times G \to G$, such that, for all $x, y, z \in G$, we have

$$2\hat{d}(m(x, y), z) \leq \left(2\hat{d}(x, z)^2 + 2\hat{d}(y, z)^2 - \hat{d}(x, y)^2\right)^{\frac{1}{2}} + 4\delta_2.$$

By Proposition 14 and its proof, there exists a vertex $m(x, y) \in G \cap p[x, y]$ such that

(5.1)

$$\left|\hat{d}(x, m(x, y)) - \frac{\hat{d}(x, y)}{2}\right| \leq \delta_1 \quad \text{and} \quad \left|\hat{d}(m(x, y), y) - \frac{\hat{d}(x, y)}{2}\right| \leq \delta_1.$$

By the definition of δ-hyperbolicity, we know that either

(1) there exists $a \in G \cap p[z, y]$ such that $d(m(x, y), a) \leq \delta + 1$, or
(2) there exists $b \in G \cap p[x, z]$ such that $d(m(x, y), b) \leq \delta + 1$.

In case (1), we have

$$\left|\hat{d}(z, m(x, y)) - \hat{d}(z, a)\right| \leq \hat{d}(m(x, y), a) \leq \delta + 1 + C_2,$$

$$\left|\hat{d}(y, m(x, y)) - \hat{d}(y, a)\right| \leq \hat{d}(m(x, y), a) \leq \delta + 1 + C_2.$$

Hence, by Proposition 10(b), we obtain

$$\begin{aligned}\hat{d}(z, m(x, y)) \quad &+\hat{d}(x, y) \leq \hat{d}(z, a) + \delta + 1 + C_2 + \hat{d}(x, y)\\ &\leq \hat{d}(z, a) + \delta + 1 + C_2 + \hat{d}(x, m(x, y)) + \hat{d}(m(x, y), y)\\ &\leq \hat{d}(z, a) + \hat{d}(a, y) + \hat{d}(x, m(x, y)) + 2\delta + 2C_2 + 2\\ &\leq \hat{d}(y, z) + \hat{d}(x, m(x, y)) + \delta',\end{aligned}$$

where $\delta' := 2\delta + 3C_2 + 2$. In case (2), we similarly have

$$\hat{d}(z, m(x, y)) + \hat{d}(x, y) \leq \hat{d}(x, z) + \hat{d}(m(x, y), y) + \delta'.$$

It follows from (5.1) that

$$\begin{aligned}\hat{d}(z, m(x, y)) + \hat{d}(x, y) &\leq \sup\left\{\hat{d}(x, z) + \hat{d}(y, m(x, y)),\ \hat{d}(y, z)\right.\\ &\qquad\qquad \left.+\hat{d}(x, m(x, y))\right\} + \delta'\\ &\leq \sup\left\{\hat{d}(x, z), \hat{d}(y, z)\right\} + \frac{\hat{d}(x, y)}{2} + \delta_1 + \delta'.\end{aligned}$$

Hence

$$2\hat{d}(z, m(x, y)) \leq 2 \sup \left\{\hat{d}(x, z), \hat{d}(y, z)\right\} - \hat{d}(x, y) + 4\delta_2, \tag{5.2}$$

where $\delta_2 := \frac{\delta_1+\delta'}{2}$.

If t, u, and v are non-negative real numbers such that $|u - v| \leq t$, then

$$(2u - v)^2 \leq 2u^2 + 2t^2 - v^2.$$

Setting $t := \inf\{\hat{d}(x, z), \hat{d}(y, z)\}$, $u := \sup\{\hat{d}(x, z), \hat{d}(y, z)\}$, $v := \hat{d}(x, y)$, we obtain

$$2 \sup \left\{\hat{d}(x, z), \hat{d}(y, z)\right\} - \hat{d}(x, y) \leq \left(2\hat{d}(x, z)^2 + 2\hat{d}(y, z)^2 - \hat{d}(x, y)^2\right)^{\frac{1}{2}}.$$

Therefore, by (5.2),

$$2\hat{d}(z, m(x, y)) \leq \left(2\hat{d}(x, z)^2 + 2\hat{d}(y, z)^2 - \hat{d}(x, y)^2\right)^{\frac{1}{2}} + 4\delta_2.$$

□

We summarize the results of this section.

Theorem 17. *Every hyperbolic group G admits a metric $\hat{d}$ with the following properties.*

(1) *$\hat{d}$ is G-invariant, i.e. $\hat{d}(g \cdot x, g \cdot y) = \hat{d}(x, y)$ for all $x, y, g \in G$.*
(2) *$\hat{d}$ is quasiisometric to the word metric d, i.e. there exist $A > 0$ and $B \geq 0$ such that*

$$\frac{1}{A}\hat{d}(x, y) - B \leq d(x, y) \leq A\hat{d}(x, y) + B$$

for all $x, y \in G$.
(3) *The metric space $(G, \hat{d})$ is weakly geodesic and strongly bolic.*

6. The Baum-Connes conjecture for hyperbolic groups

In this section, we combine Theorem 17 with Lafforgue's work to prove the main result of this paper.

Definition 18. An action of a topological group G on a topological space X is called proper if the map $G \times X \to X \times X$ given by $(g, x) \mapsto (x, gx)$ is a proper map, that is the preimages of compact subsets are compact.

When G is discrete, an action is proper iff it is properly discontinuous, i.e. if the set $\{g \in G \mid K \cap gK \neq \emptyset\}$ is finite for any compact $K \subseteq X$.

The following deep theorem was proved by Lafforgue using Banach KK-theory.

Theorem 19 (Lafforgue [13]). *If a discrete group G has property RD, and G acts properly and isometrically on a strongly bolic, weakly geodesic, and uniformly locally finite metric space, then the Baum-Connes conjecture holds for G.*

Theorem 20. *The Baum-Connes conjecture holds for hyperbolic groups and their subgroups.*

Proof. Let H be a subgroup of a hyperbolic group G. By Theorem 17(2), there exist constants $A > 0$ and $B \geq 0$ such that $d(a, b) \leq A\,\hat{d}(a, b) + B$ for all $a, b \in G$. Hence $(G, \hat{d})$ is uniformly locally finite and the H-action on $(G, \hat{d})$ is proper. By Theorem 17, $(G, \hat{d})$ is weakly geodesic and strongly bolic, and the H-action on $(G, \hat{d})$ is isometric. By a theorem of P. de la Harpe and P. Jolissaint, H has property RD [5,9]. Now Theorem 19 implies Theorem 20. □

Theorem 20 has been proved independently by Vincent Lafforgue using a different and elegant method [14].

The following result is a direct consequence of Theorem 20.

Theorem 21. *The Kadison-Kaplansky conjecture holds for any torsion free subgroup G of a hyperbolic group, i.e. there exists no non-trivial projection in the reduced group C^*-algebra $C_r^*(G)$.*

Recall that an element p in $C_r^*(G)$ is said to be a projection if $p^* = p$, $p^2 = p$. A projection in $C_r^*(G)$ is said to be non-trivial if $p \neq 0, 1$. It is well known that the Baum-Connes conjecture for a torsion free discrete group G implies the Kadison-Kaplansky conjecture for G [3,2].

Michael Puschnigg has independently proved Theorem 21 using a beautiful local cyclic homology method [17]. Ronghui Ji has previously proved that there exists no non-trivial idempotent in the Banach algebra $\ell^1(G)$ for any torsion free hyperbolic group [8].

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