SUBMULTIPLICATIVITY AND THE HANNA NEUMANN CONJECTURE

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Abstract. We define *submultiplicativity* of ℓ^2-numbers in the category of Γ-complexes over a given Γ-complex $\hat{X}$, which generalizes the statement of the Strengthened Hanna Neumann Conjecture (SHNC). In the case when Γ is a left-orderable group and $\hat{X}$ is a free Γ-complex, we prove submultiplicativity for the subcategory consisting of Γ-ordered *leafages* over $\hat{X}$ with an additional analytic assumption called the *deep-fall property*. We show that the deep-fall property is satisfied for graphs. This implies SHNC.

うらやまし　美しう成て　散る紅葉

各務支考

How I envy maple leafage
which turns beautiful
then falls

Kagami Shikō

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1. Introduction.

The Hanna Neumann Conjecture (HNC) [15, 16] can be stated as follows.

Conjecture (HNC)**.** *Suppose* Γ *is a free group and* A *and* B *are its finitely generated subgroups. Then*

$$\bar{r}(A \cap B) \leq \bar{r}(A)\, \bar{r}(B).$$

Here $\bar{r}(\Gamma) := \max\{0, \operatorname{rk}\Gamma - 1\}$ is the *reduced rank* of a free group Γ, introduced by Walter Neumann [17, p. 162] and named so by Dicks [2, p. 373]. Walter Neumann [17, p.164] further proposed the following Strengthened Hanna Neumann Conjecture (SHNC). Let $A\backslash\Gamma/B$ be the set of all double cosets AgB for $g \in \Gamma$ and $s : A\backslash\Gamma/B \to \Gamma$ be a section of the quotient map $\Gamma \to A\backslash\Gamma/B$. Denote $A^z := z^{-1}Az$.

Conjecture (SHNC)**.** *Suppose* Γ *is a free group and* A *and* B *are its finitely generated subgroups. Then*

$$\sum_{z \in s(A\backslash\Gamma/B)} \bar{r}(A^z \cap B) \leq \bar{r}(A)\, \bar{r}(B).$$

Let Γ be a free group, X be a graph with fundamental group Γ, and A and B be finitely generated subgroups of Γ. Stallings [21] showed that A and B can be realized by immersions $Y \to X$ and $Z \to X$ of finite graphs, and that $A \cap B$ is realized by a connected component of their fiber product

(1)
$$\begin{tikzcd} S \arrow[r, "\nu"] \arrow[d, "\mu"'] & Z \arrow[d, "\beta"] \\ Y \arrow[r, "\alpha"] & X. \end{tikzcd}$$

Gersten [7] further refined that approach to give a graph-theoretic (and simpler) proof of Hanna Neuman's original upper bound:

$$\bar{r}(A \cap B) \leq 2\, \bar{r}(A)\, \bar{r}(B).$$

Systems of complexes defined in [14, subsection 3.1] are certain diagrams obtained as multiple pull-backs (see 4.2 below). Systems consisting of graphs incorporate Stallings' fiber product diagrams (1). They provide Γ-equivariant versions of (1), allow restating SHNC in terms of ℓ^2-Betti numbers and generalizing the statement of SHNC (see [14]). It was first observed by Warren Dicks that HNC can be restated in terms of the first ℓ^2-Betti numbers of A, B and $A \cap B$.

In this paper we discuss *submultiplicativity* which is the term for "a general SHNC-like property for Γ-complexes". The precise definition is given in 2.4 below. We work with complexes of arbitrary dimension whenever possible, since this is more general and requires no additional effort. First we provide some general constructions and conditions that imply submultiplicativity, notably left-invariant orders in 3.1, *leafages* in 4.1, and the *deep-fall property* in 5.2. Theorem 14 is the main result of this paper which lists the necessary conditions in the generality of complexes. Then we show that those conditions are satisfied in the case of graphs (section 6); this implies SHNC. A reader interested strictly in the proof of SHNC should always think of graphs and dimension $i = 1$ whenever the word "complex" comes up.

For previous results related to HNC the reader is referred to Burns [1], Imrich [8], Servatius [19], Stallings [21], Gersten [7], Nickolas [18], Walter Neumann [17], Feuerman [5], Tardos [22], Dicks [2], Dicks-Formanek [3], Khan [11], Meakin-Weil [13], Sergei Ivanov [9, 10], Dicks-Ivanov [4].

A proof of SHNC has also been announced in a recent preprint by Joel Friedman [6].

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2. Complexes and submultiplicativity.

2.1. **Complexes and graphs.** In this paper, by a *complex* we will mean a cell complex. Maps between cell complexes will be assumed to be *combinatorial*, meaning that they send open cells homeomorphically onto open cells. We require that cells in cell complexes are oriented, and that actions on cell complexes preserve the orientations of cells. Each complex Y will be formally viewed as a disjoint union of its cells $\Sigma_*^Y := \bigsqcup_{i \geq 0} \Sigma_i^Y$, where Σ_i^Y is the set of i-cells in Y.

A *graph* is a 1-dimensional cell complex. The orientation on the edges of a graph allows assigning to each edge σ its initial and terminal vertices, denoted σ^- and σ^+, respectively.

Let Γ be a group. A Γ-complex $\hat{Y}$ is *of type* $\mathcal{F}$ if the quotient $\Gamma \backslash \hat{Y}$ is finite. If $\hat{Y}$ is a Γ-complex of type $\mathcal{F}$, then the boundary maps $\partial : \mathbb{C}\Sigma_i^{\hat{Y}} \to \mathbb{C}\Sigma_{i-1}^{\hat{Y}}$ extend to a bounded map of Γ-modules $\partial : \ell^2(\Sigma_i^{\hat{Y}}) \to \ell^2(\Sigma_{i-1}^{\hat{Y}})$, i.e. a morphism of Hilbert Γ-modules. Combining all the above maps ∂ into one map defines the total boundary operator $\partial : \ell^2(\hat{Y}) \to \ell^2(\hat{Y})$. More generally, this operator is defined if $\hat{Y}$ is a uniformly locally finite complex.

2.2. **Complexes over** $(\hat{X}, \Gamma)$**.** Let $\hat{X}$ be a Γ-complex. A *complex over* $(\hat{X}, \Gamma)$ is a pair $(\hat{Y}, \hat{\alpha})$, where $\hat{Y}$ is a Γ-complex and $\hat{\alpha} : \hat{Y} \to \hat{X}$ is a Γ-equivariant map. For simplicity we will just say in this case that "$\hat{Y}$ is a complex over $(\hat{X}, \Gamma)$" or "$\hat{\alpha}$ is a complex over $(\hat{X}, \Gamma)$".

The complexes over $(\hat{X}, \Gamma)$ form a category $\mathrm{Compl}(\hat{X}, \Gamma)$, where morphisms $(\hat{Y}, \hat{\alpha}) \to (\hat{Y}', \hat{\alpha}')$ are Γ-equivariant maps $\varphi : \hat{Y} \to \hat{Y}'$ such that $\alpha' \circ \varphi = \alpha$.

The *product* of two complexes $\hat{Y}$ and $\hat{Z}$ in over $(\hat{X}, \Gamma)$ is the fiber product over $\hat{X}$, denoted $\hat{Y} \,\hat{\square}\, \hat{Z}$:

$$\begin{array}{ccc} \hat{Y} \,\hat{\square}\, \hat{Z} & \xrightarrow{\hat{\nu}} & \hat{Z} \\ {\scriptstyle \hat{\mu}}\downarrow & & \downarrow{\scriptstyle \hat{\beta}} \\ \hat{Y} & \xrightarrow{\hat{\alpha}} & \hat{X}. \end{array}$$

2.3. ℓ^2**-numbers.** For a detailed exposition of Hilbert modules and Murray-von Neumann dimension see [12]. In this paper we will only use the standard facts that are collected in [14, subsections 4.1 and 4.2].

For a Γ-complex $\hat{Y}$ of type $\mathcal{F}$ define

$$a_i^{(2)}(\hat{Y},\Gamma) := \dim_\Gamma \operatorname{Ker}\Big(\partial : \ell^2(\Sigma_i^{\hat{Y}}) \to \ell^2(\Sigma_{i-1}^{\hat{Y}})\Big),$$
$$b_i^{(2)}(\hat{Y},\Gamma) := \dim_\Gamma H_i^{(2)}(\hat{Y}),$$

where $\dim_\Gamma$ is the Murray-von Neumann dimension of Hilbert Γ-modules. $b_i^{(2)}(\hat{Y};\Gamma)$ is called the ith *ℓ^2-Betti number*. It is possible to define these numbers without the type $\mathcal{F}$ assumption; we omit the details.

2.4. **Submultiplicativity.** Let Γ be a group, $\hat{X}$ be a Γ-complex, $\hat{Y}$ and $\hat{Z}$ be complexes over $(\hat{X},\Gamma)$, and i be an integer. We pose the following general questions.

(a) Under what conditions

$$a_i^{(2)}(\hat{Y}\hat{\Box}\hat{Z},\Gamma) \leq a_i^{(2)}(\hat{Y},\Gamma)\cdot a_i^{(2)}(\hat{Z},\Gamma)\ ?$$

(b) Under what conditions

$$b_i^{(2)}(\hat{Y}\hat{\Box}\hat{Z},\Gamma) \leq b_i^{(2)}(\hat{Y},\Gamma)\cdot b_i^{(2)}(\hat{Z},\Gamma)\ ?$$

These properties might be called *a-submultiplicativity* and *b-submultiplicativity*, respectively. In the case when $\hat{Y}$ is a graph and $i=1$, the two numbers agree:

$$a_1^{(2)}(\hat{Y},\Gamma) = b_1^{(2)}(\hat{Y},\Gamma),$$

so the two questions are equivalent. SHNC is equivalent to submultiplicativity in dimension 1 for certain appropriately chosen graphs $\hat{X}$, $\hat{Y}$, $\hat{Z}$ (see [14]), so the above submultiplicativity questions generalize SHNC.

3. Complexes and orders.

3.1. **Law and order.** In what follows, we will clearly distinguish between ordered Γ-complexes and Γ-ordered complexes.

Given a complex $\hat{X}$, by an *order on* $\hat{X}$ we will mean a choice of a total order $\leq$ on each $\Sigma_i^{\hat{X}}$.

Let Γ be any group. $\hat{X}$ will be called an *ordered Γ-complex* if

- $\hat{X}$ is a Γ-complex and
- for each i, $\Sigma_i^{\hat{X}}$ is given a Γ-invariant total order $\leq$.

Now additionally assume that Γ is left-ordered, meaning that there is a total order $\leq$ on Γ such that $a\leq b$ implies $ga\leq gb$ for all $a,b,g\in\Gamma$.

Suppose $\hat{X}$ is a free Γ-complex and $\bar{\Sigma}_i^{\hat{X}}\subseteq\Sigma_i^{\hat{X}}$ is a fundamental domain for the free Γ-action on $\Sigma_i^{\hat{X}}$. Then $\Sigma_i^{\hat{X}}=\Gamma\bar{\Sigma}_i^{\hat{X}}$ and the map $\Gamma\times\bar{\Sigma}_i^{\hat{X}}\to\Sigma_i^{\hat{X}}$ given by $(g,\bar{\sigma})\mapsto g\bar{\sigma}$ is a bijection.

Fix any order on the fundamental domain $\bar{\Sigma}_i^{\hat{X}}$. Put an order on $\Sigma_i^{\hat{X}}$ by identifying it with $\Gamma\times\bar{\Sigma}_i^{\hat{X}}$ as above and taking the lexicographic order. Specifically, the order is defined according to the law

$$g\bar{\sigma} < h\bar{\tau} \quad\Leftrightarrow\quad g<h \quad\text{or}\quad (g=h \text{ and } \bar{\sigma}<\bar{\tau}) \tag{2}$$

for $g\bar{\sigma},h\bar{\tau}\in\Gamma\bar{\Sigma}_i^{\hat{X}}=\Sigma_i^{\hat{X}}$. Since the order on Γ is left-invariant, we immediately obtain

Lemma 1. *The above order on $\Sigma_i^{\hat{X}}$ is (left) Γ-invariant.*

We will say that $\hat{X}$ is a Γ-*ordered complex* if

- Γ is a left-ordered group and
- $\hat{X}$ is a *free* Γ-complex with an order induced on each $\Sigma_i^{\hat{X}}$ as above.

Each Γ-ordered complex is an ordered Γ-complex, but the converse is false in general.

3.2. **Cones and maps.** For a partially ordered set $(T, \leq)$ and $t \in T$, denote

- $[T < t] := \{s \in T \mid s < t\}$,
- $[T \leq t] := \{s \in T \mid s \leq t\}$.

These are the *negative cones* of the order on T.

A function $\varphi : S \to T$ between partially ordered sets is called *order-preserving* if for all $s, s' \in S$, $s \leq s'$ implies $\varphi(s) \leq \varphi(s')$. Equivalently, if $s < s'$ implies $\varphi(s) \leq \varphi(s')$. The function $\varphi : S \to T$ is called *strictly order-preserving* if for all $s, s' \in S$, $s < s'$ implies $\varphi(s) < \varphi(s')$.

If $(T, \leq)$ is a partially ordered set with a left action by a group Γ preserving the order $\leq$, then Γ also preserves the strict order $<$ and sends cones to cones. Specifically, $g \in \Gamma$,

$$g[T \leq t] = [T \leq gt] \quad \text{and} \quad g[T < t] = [T < gt].$$

3.3. **Pull-back orders.** Let T be a set, $(S, \leq)$ be a totally ordered set, and $\alpha : T \to S$ be a function. Put any total order on each fiber $\alpha^{-1}(s)$, $s \in S$. The union of these orders is a partial order on T which we denote $\leq_{\text{fib}}$. For $t, t' \in T$, set

$$\begin{aligned} t <_{\text{fib}} t' \quad &\Leftrightarrow \quad t \leq_{\text{fib}} t' \quad \text{and} \quad t \neq t', \\ t < t' \quad &\Leftrightarrow \quad \alpha(t) < \alpha(t') \quad \text{or} \quad t <_{\text{fib}} t', \\ t \leq t' \quad &\Leftrightarrow \quad t < t' \quad \text{or} \quad t = t'. \end{aligned} \tag{3}$$

A total order $\leq$ on T defined in this way will be called *a pull-back order* on T.

Any Γ-ordering on a complex $\hat{X}$ as in 3.1 can be viewed as a special case of pull-back order – one corresponding to a Γ-equivariant map $\Sigma_i^{\hat{X}} \to \Gamma$.

Let $\hat{X}$ be a Γ-ordered complex and $\hat{\alpha} : \hat{Y} \to \hat{X}$ is a complex over $(\hat{X}, \Gamma)$. Then $\hat{Y}$ can be given a Γ-invariant order as follows. For the fundamental domain $\bar{\Sigma}_i^{\hat{X}}$ of $\Sigma_i^{\hat{X}}$, denote $\bar{\Sigma}_i^{\hat{Y}} := \hat{\alpha}^{-1}(\bar{\Sigma}_i^{\hat{X}})$, then $\bar{\Sigma}_i^{\hat{Y}}$ is a fundamental domain of $\Sigma_i^{\hat{Y}}$. Take any pull-back order on $\bar{\Sigma}_i^{\hat{Y}}$ via $\hat{\alpha} : \bar{\Sigma}_i^{\hat{Y}} \to \bar{\Sigma}_i^{\hat{X}}$, then use it to define an order on $\Sigma_i^{\hat{Y}}$ as in (2). This turns $\hat{Y}$ into a Γ-ordered complex. When the order on Y is obtained in this way, we will say that $\hat{\alpha} : \hat{Y} \to \hat{X}$ is a Γ-*ordered complex over* $\hat{X}$. Note that this order on $\hat{Y}$ is also an example of a pull-back order via $\hat{\alpha} : \hat{Y} \to \hat{X}$ as defined in (3). Note also that $\hat{\alpha}$ is order-preserving, but not necessarily strictly order-preserving.

3.4. **Order-essential and order-inessential cells.**

Definition 2. *Let $\hat{Y}$ be an ordered Γ-complex. A cell $\sigma \in \Sigma_i^{\hat{Y}}$ will be called* order-essential *if any of the following equivalent conditions holds:*

(a) $\partial\sigma \in \overline{\partial(\ell^2[\Sigma_i^{\hat{Y}} < \sigma])}$ *in* $\ell^2(\Sigma_{i-1}^{\hat{Y}})$,
(b) $\overline{\partial(\ell^2[\Sigma_i^{\hat{Y}} \leq \sigma])} \subseteq \overline{\partial(\ell^2[\Sigma_i^{\hat{Y}} < \sigma])}$ *in* $\ell^2(\Sigma_{i-1}^{\hat{Y}})$,

(c) $\overline{\partial(\ell^2[\Sigma_i^{\hat{Y}} \leq \sigma])} = \overline{\partial(\ell^2[\Sigma_i^{\hat{Y}} < \sigma])}$ *in* $\ell^2(\Sigma_{i-1}^{\hat{Y}})$,

and order-inessential *otherwise. Denote* $\mathbb{E}_i^{\hat{Y}}$ *and* $\mathbb{I}_i^{\hat{Y}}$ *the sets of order-essential and order-inessential cells in* $\hat{Y}$*, respectively.*

Here $\ell^2[\Sigma_i^{\hat{Y}} < \sigma]$ denotes the Hilbert space with basis $[\Sigma_i^{\hat{Y}} < \sigma]$, viewed as a subspace of $\ell^2(\Sigma_i^{\hat{Y}})$. The bar represents the closure in $\ell^2(\Sigma_{i-1}^{\hat{Y}})$. If condition (a) holds, we will say that σ *falls* into its open negative cone. By definition, $\Sigma_i^{\hat{Y}} = \mathbb{E}_i^{\hat{Y}} \sqcup \mathbb{I}_i^{\hat{Y}}$.

Lemma 3. *The sets* $\mathbb{E}_i^{\hat{Y}}$ *and* $\mathbb{I}_i^{\hat{Y}}$ *are* Γ*-invariant.*

Proof. For any $g \in \Gamma$,

$$g\sigma \in \mathbb{E}_i^{\hat{Y}} \quad \Leftrightarrow \quad \partial(g\sigma) \in \overline{\partial(\ell^2[\Sigma_i^{\hat{Y}} < g\sigma])} \quad \Leftrightarrow \quad g(\partial\sigma) \in g\Big(\overline{\partial(\ell^2[\Sigma_i^{\hat{Y}} < \sigma])}\Big)$$

$$\Leftrightarrow \quad \partial\sigma \in \overline{\partial(\ell^2[\Sigma_i^{\hat{Y}} < \sigma])} \quad \Leftrightarrow \quad \sigma \in \mathbb{E}_i^{\hat{Y}},$$

and similarly for $\mathbb{I}_i^{\hat{Y}}$. □

Definition 4. *Suppose that* $\hat{Y}$ *is a* Γ*-ordered complex,* $Y := \Gamma\backslash\hat{Y}$*, and* $p_Y : \hat{Y} \to Y$ *be the quotient map. A cell* $\sigma \in \Sigma_i^Y$ *will be called* order-essential *if any preimage of* σ *under* $p_Y : \hat{Y} \to Y$ *is order-essential. Similarly, it is* order-inessential *if any preimage of* σ *is order-inessential. Denote* $\mathbb{E}_i^Y$ *and* $\mathbb{I}_i^Y$ *the sets of order-essential and order-inessential edges in* Σ_i^Y*, respectively.*

$\mathbb{E}_i^Y$ and $\mathbb{I}_i^Y$ depend on the choice of $\hat{Y}$ and the order on $\hat{Y}$, but we suppress them in the notation. Since the Γ-action is free, the Γ-orbits of cells in $\hat{Y}$ are exactly the preimages of cells in Y, so Lemma 3 guarantees that $\Sigma_i^Y = \mathbb{E}_i^Y \sqcup \mathbb{I}_i^Y$.

4. Leafages and systems.

4.1. **Leafages.** As defined in [14], a *leafage* is a map $\hat{\alpha} : \hat{Y} \to \hat{Z}$ between complexes whose restriction to each connected component is injective. A Γ*-leafage* is a leafage $\hat{\alpha} : \hat{Y} \to \hat{X}$ in which $\hat{Y}$ and $\hat{X}$ are given left Γ-actions that commute with $\hat{\alpha}$. A Γ*-ordered leafage* is a Γ-leafage $\hat{\alpha} : \hat{Y} \to \hat{Z}$ in which $\hat{X}$ and $\hat{X}$ are Γ-ordered complexes and the order on $\hat{Y}$ is a pull-back order from $\hat{X}$ as in 3.3.

Given a free Γ-complex $\hat{X}$, the Γ-leafages over $\hat{X}$ form a category $\mathrm{Leaf}(\hat{X},\Gamma)$: objects are Γ-leafages over $\hat{X}$ and morphisms are Γ-equivariant maps $\hat{Y} \to \hat{Y}'$ compatible with the maps to $\hat{X}$. $\mathrm{Leaf}(\hat{X},\Gamma)$ is a full subcategory of $\mathrm{Compl}(\hat{X},\Gamma)$.

Similarly, given a Γ-ordered complex $\hat{X}$, the Γ-ordered leafages over $\hat{X}$ form a category $\mathrm{Leaf}(\hat{X},\Gamma)^{\leq}$: the objects are Γ-ordered leafages over $\hat{X}$ and morphisms are the same as in $\mathrm{Leaf}(\hat{X},\Gamma)$. We *do not* require morphisms in $\mathrm{Leaf}(\hat{X},\Gamma)^{\leq}$ to be order-preserving.

A product of two objects in this category is defined, but not uniquely. By *a product* we will mean the usual fiber product together with *some* choice of a pull-back order. As the term suggests, a fiber in the fiber product is the product of fibers. So if necessary, one can make the order on the product canonical by putting the lexicographic order on each of its fibers.

The following lemma is immediate.

Lemma 5. *Suppose $\hat{\alpha} : \hat{Y} \to \hat{X}$ and $\hat{\beta} : \hat{Z} \to \hat{X}$ are complexes over $\hat{X}$ and*

$$\begin{tikzcd} \hat{S} \arrow[r, "\hat{\nu}"] \arrow[d, "\hat{\mu}"'] & \hat{Z} \arrow[d, "\hat{\beta}"] \\ \hat{Y} \arrow[r, "\hat{\alpha}"] & \hat{X} \end{tikzcd} \tag{4}$$

is their product diagram.

(a) *If $\hat{\alpha}$ is a leafage, then $\hat{\nu}$ is a leafage.*
(b) *If $\hat{\alpha}$ and $\hat{\beta}$ are leafages, then $\hat{\alpha} \circ \hat{\mu}$ is a leafage.*

Lemma 6. *Suppose $\hat{\alpha} : \hat{Y} \to \hat{X}$ is a Γ-ordered leafage. Then for each i and each component K of $\hat{Y}$, the restriction of α to Σ_i^K is strictly order-preserving.*

Proof. Take any $\sigma, \tau \in \Sigma_i^K$ and assume $\sigma < \tau$. Since $\hat{\alpha}$ is a leafage, then $\hat{\alpha}(\sigma) \neq \hat{\alpha}(\tau)$. Suppose $\hat{\alpha}(\tau) < \hat{\alpha}(\sigma)$, then by the definition of the pull-back order on $\hat{Y}$, $\tau < \sigma$, which is a contradiction. Hence $\hat{\alpha}(\sigma) < \hat{\alpha}(\tau)$. □

The following lemma is an easy exercise.

Lemma 7. *Let $\hat{Y}$ be a Γ-complex of type $\mathcal{F}$ and K be a connected component of $\hat{Y}$. Then the orthogonal projection $p_K : \ell^2(\hat{Y}) \to \ell^2(K)$ commutes with the boundary operator $\partial : \ell^2(\hat{Y}) \to \ell^2(\hat{Y})$.*

Lemma 8. *Let $\hat{Y}$ be a Γ-complex of type $\mathcal{F}$, $\sigma \in \Sigma_i^{\hat{Y}}$, and K be the connected component of $\hat{Y}$ containing σ. Then $\partial\sigma \in \overline{\partial(\ell^2[\Sigma_i^{\hat{Y}} < \sigma])}$ if and only if $\partial\sigma \in \overline{\partial(\ell^2[\Sigma_i^K < \sigma])}$.*

Proof. The "if" direction is clear. For "only if", use Lemma 7:

$$\begin{aligned} &\partial\sigma \in \partial(p_K(\sigma)) = p_K(\partial\sigma) \in p_K\Big(\overline{\partial(\ell^2[\Sigma_i^{\hat{Y}} < \sigma])}\Big) \\ &\subseteq \overline{p_K(\partial(\ell^2[\Sigma_i^{\hat{Y}} < \sigma]))} = \overline{\partial(p_K(\ell^2[\Sigma_i^{\hat{Y}} < \sigma]))} \\ &= \overline{\partial(\ell^2([\Sigma_i^{\hat{Y}} < \sigma] \cap \Sigma_i^K))} = \overline{\partial(\ell^2[\Sigma_i^K < \sigma])}. \end{aligned}$$

□

Lemma 9 (Leafage maps preserve order-essential cells)**.** *Suppose $\hat{X}$ is a Γ-ordered complex, $\hat{\alpha} : \hat{Y} \to \hat{X}$ and $\hat{\alpha}' : \hat{Y}' \to \hat{X}$ are Γ-ordered leafages of type $\mathcal{F}$ over $\hat{X}$, and $\lambda : \hat{Y} \to \hat{Y}'$ is a morphism of Γ-leafages over $\hat{X}$. Then the following hold.*

(a) *For all i, $\hat{\lambda}(\mathbb{E}_i^{\hat{Y}}) \subseteq \mathbb{E}_i^{\hat{Y}'}$.*
(b) *If $Y := \Gamma\backslash\hat{Y}$, $Y' := \Gamma\backslash\hat{Y}'$, and $\lambda : Y \to Y'$ is induced by $\hat{\lambda}$, then $\lambda(\mathbb{E}_i^Y) \subseteq \mathbb{E}_i^{Y'}$.*

Proof. **(a)** Take any $\sigma \in \mathbb{E}_i^{\hat{Y}}$ and let K be the component of $\hat{Y}$ containing σ. By Lemma 8, $\partial\sigma \in \overline{\partial(\ell^2[\Sigma_i^K < \sigma])}$. Let K' be the connected component of Y' containing $\hat{\lambda}(K)$. By Lemma 6, the restricted maps $\hat{\alpha} : \Sigma_i^K \to \Sigma_i^{\hat{X}}$ and $\hat{\alpha}' : \Sigma_i^{K'} \to \Sigma_i^{\hat{X}}$ are strictly order-preserving, hence the restricted map $\hat{\lambda} : \Sigma_i^K \to \Sigma_i^{K'} \subseteq \Sigma_i^{\hat{Y}'}$ is strictly order-preserving. Therefore

$$\partial(\hat{\lambda}(\sigma)) = \hat{\lambda}(\partial\sigma) \in \hat{\lambda}\left(\overline{\partial(\ell^2[\Sigma_i^K < \sigma])}\right) \subseteq \overline{\partial(\ell^2(\hat{\lambda}([\Sigma_i^K < \sigma])))} \subseteq \overline{\partial(\ell^2[\Sigma_i^{\hat{Y}'} < \hat{\lambda}(\sigma)])},$$

so $\hat{\lambda}(\sigma) \in \mathbb{E}_i^{\hat{Y}'}$.

(b) clearly follows from (a) by the definition of $\mathbb{E}_i^Y$ and $\mathbb{E}_i^{Y'}$. □

4.2. **Systems.** Systems of complexes were introduced in [14, subsection 3.1] to generalize the statement of SHNC. Systems provide concrete examples of leafages and their products.

Let Γ be a group. A Γ-*system* is a diagram as in (5) obtained as follows.

(5)
$$\mathcal{S} : \quad \begin{tikzcd}
S \arrow[rr, "\nu"] \arrow[dd, "\mu"'] & & Z \arrow[dd, "\beta"] & \\
& \hat{S} \arrow[ul, "p_S"'] \arrow[rr, "\hat{\nu}"] \arrow[dd, "\hat{\mu}"'] & & \hat{Z} \arrow[ul, "p_Z"'] \arrow[dd, "\hat{\beta}"] \\
Y \arrow[rr, "\alpha"] & & X & \\
& \hat{Y} \arrow[ul, "p_Y"] \arrow[rr, "\hat{\alpha}"] & & \hat{X}. \arrow[ul, "p_X"']
\end{tikzcd}$$

Start with any cell complex $\hat{X}$ with a free Γ-action and let X be the quotient $\Gamma\backslash\hat{X}$. Denote $p_X : \hat{X} \to X$ the quotient map. Let $\alpha : Y \to X$ and $\beta : Z \to X$ be *immersions*, defined as maps of complexes that can be extended to (not necessarily finite) covers of X. Let

(6)
$$\begin{tikzcd}
S \arrow[r, "\nu"] \arrow[d, "\mu"'] & Z \arrow[d, "\beta"] \\
Y \arrow[r, "\alpha"] & X
\end{tikzcd}$$

be the fiber-product diagram for Y and Z. Now the diagram (5) is defined to be the pull-back of the whole diagram (6) under $p_X : \hat{X} \to X$. It is called the *system generated by* α*,* β *and* p_X. For this general definition, none of the complexes in the system is assumed to be finite or connected or simply connected.

Stallings [21] defined immersions of graphs as locally injective maps. It also can be deduced from the arguments in [21] that immersions of finite graphs are exactly the maps of finite graphs that can be extended to finite covers, so the above definition of immersions generalizes this notion to (finite or infinite) complexes.

If in a system $\mathcal{S}$ the map $p_X : \hat{X} \to X$ is the universal cover of X, then $\hat{Y} \to \hat{X}$ and $\hat{Z} \to \hat{X}$ are leafages, and their product $\hat{S} \to \hat{X}$ is as well (see [14, Theorem 7(c)] and Lemma 5 above).

A system will be called Γ-*ordered* if $\hat{X}$ is Γ-ordered and $\hat{Y}$, $\hat{Z}$ and $\hat{S}$ are given (Γ-invariant) pull-back orders by $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\alpha} \circ \hat{\mu}$, respectively, as in 3.3.

5. Deep fall and finite fall.

5.1. Cones and Hilbert spaces.

Proposition 10. *For each Γ-ordered complex $\hat{Y}$,*

$$\dim_\Gamma \overline{\partial(\ell^2(\mathbb{I}_i^{\hat{Y}}))} = \dim_\Gamma \ell^2(\mathbb{I}_i^{\hat{Y}}).$$

Equivalently, the restriction of the boundary operator to $\ell^2(\mathbb{I}_i^{\hat{Y}})$, $\partial\colon \ell^2(\mathbb{I}_i^{\hat{Y}}) \to \ell^2(\Sigma_{i-1}^{\hat{Y}})$, is an injective morphism of Hilbert Γ-modules.

Proof. For each $\sigma \in \mathbb{I}_i^{\hat{Y}}$, $\partial\sigma \notin \overline{\partial(\ell^2[\Sigma_i^{\hat{Y}} < \sigma])}$. In particular, $\partial\sigma \notin \overline{\partial(\ell^2[\mathbb{I}_i^{\hat{Y}} < \sigma])}$. This implies that

$$V_\sigma := (\partial(\ell^2[\mathbb{I}_i^{\hat{Y}} < \sigma]))^\perp \cap \overline{\partial(\ell^2[\mathbb{I}_i^{\hat{Y}} \leq \sigma])}$$

is a one-dimensional subspace of $\overline{\partial(\ell^2(\mathbb{I}_i^{\hat{Y}}))}$. Pick a unit vector e_σ in V_σ. For any $g \in \Gamma$, $V_{g\sigma} = gV_\sigma$. Therefore we can pick e_σ in an equivariant fashion so that for all $g \in \Gamma$ and $\sigma \in \mathbb{I}_i^{\hat{Y}}$, $e_{g\sigma} = ge_\sigma$.

If $\sigma, \tau \in \mathbb{I}_i^{\hat{Y}}$ and $\sigma < \tau$, then

$$e_\tau \in (\partial(\ell^2[\mathbb{I}_i^{\hat{Y}} < \tau]))^\perp \quad \text{and} \quad e_\sigma \in \overline{\partial(\ell^2[\mathbb{I}_i^{\hat{Y}} \leq \sigma])} \subseteq \overline{\partial(\ell^2[\mathbb{I}_i^{\hat{Y}} < \tau])},$$

hence $e_\sigma \perp e_\tau$. Since the order on $\Sigma_i^{\hat{Y}}$ is total, this implies that for all $\sigma, \tau \in \mathbb{I}_i^{\hat{Y}}$,

$$\sigma \neq \tau \quad \Rightarrow \quad e_\sigma \perp e_\tau.$$

Then $\{e_\sigma \mid \sigma \in \mathbb{I}_i^{\hat{Y}}\}$ is an orthonormal subset of $\overline{\partial(\ell^2(\mathbb{I}_i^{\hat{Y}}))}$, and the map

$$\mathbb{I}_i^{\hat{Y}} \to \overline{\partial(\ell^2(\mathbb{I}_i^{\hat{Y}}))}, \quad \sigma \mapsto e_\sigma,$$

is Γ-equivariant and extends to an isometric embedding of Hilbert Γ-modules

$$\varphi : \ell^2(\mathbb{I}_i^{\hat{Y}}) \hookrightarrow \overline{\partial(\ell^2(\mathbb{I}_i^{\hat{Y}}))}.$$

This implies $\dim_\Gamma \ell^2(\mathbb{I}_i^{\hat{Y}}) \leq \dim_\Gamma \overline{\partial(\ell^2(\mathbb{I}_i^{\hat{Y}}))}$. The converse inequality $\dim_\Gamma \ell^2(\mathbb{I}_i^{\hat{Y}}) \geq \dim_\Gamma \overline{\partial(\ell^2(\mathbb{I}_i^{\hat{Y}}))}$ holds by the additivity of dimension for the weakly exact sequence

$$0 \to \mathrm{Ker} \xrightarrow{\subseteq} \ell^2(\mathbb{I}_i^{\hat{Y}}) \xrightarrow{\partial} \overline{\partial(\ell^2(\mathbb{I}_i^{\hat{Y}}))} \to 0,$$

where Ker is the kernel of the map $\partial : \ell^2(\mathbb{I}_i^{\hat{Y}}) \to \partial(\ell^2(\mathbb{I}_i^{\hat{Y}})) \subseteq \ell^2(\Sigma_i^{\hat{Y}})$. Therefore $\dim_\Gamma \ell^2(\mathbb{I}_i^{\hat{Y}}) = \dim_\Gamma \overline{\partial(\ell^2(\mathbb{I}_i^{\hat{Y}}))}$ and $\mathrm{Ker} = 0$. Conversely, if $\mathrm{Ker} = 0$ then the dimensions are equal. $\square$

5.2. **The deep-fall property.** Let $\hat{Y}$ be a Γ-ordered complex of type $\mathcal{F}$ and $i \geq 0$. $\hat{Y}$ will be called *deep-fall*, or *i-deep-fall*, if for any $\sigma \in \mathbb{E}_i^{\hat{Y}}$,

$$\partial\sigma \in \overline{\partial(\ell^2[\mathbb{I}_i^{\hat{Y}} < \sigma])}.$$

Theorem 11. *If $\hat{Y}$ is a deep-fall Γ-ordered complex then*

$$\overline{\partial(\ell^2(\Sigma_i^{\hat{Y}}))} = \overline{\partial(\ell^2(\mathbb{I}_i^{\hat{Y}}))} \quad \textit{in} \quad \ell^2(\Sigma_{i-1}^{\hat{Y}}).$$

Proof. It suffices to to prove the inclusion "$\subseteq$". Take any $\sigma \in \mathbb{E}_i^{\hat{Y}}$. The deep-fall property implies that $\partial\sigma \in \overline{\partial(\ell^2[\mathbb{I}_i^{\hat{Y}} < \sigma])} \subseteq \overline{\partial(\ell^2(\mathbb{I}_i^{\hat{Y}}))}$. Thus $\partial(\ell^2(\mathbb{E}_i^{\hat{Y}})) \subseteq \overline{\partial(\ell^2(\mathbb{I}_i^{\hat{Y}}))}$ and

$$\overline{\partial(\ell^2(\Sigma_i^{\hat{Y}}))} \subseteq \overline{\partial(\ell^2(\mathbb{E}_i^{\hat{Y}}) + \ell^2(\mathbb{I}_i^{\hat{Y}}))} \subseteq \overline{\partial(\ell^2(\mathbb{E}_i^{\hat{Y}})) + \partial(\ell^2(\mathbb{I}_i^{\hat{Y}}))} \subseteq \overline{\partial(\ell^2(\mathbb{I}_i^{\hat{Y}}))}.$$

□

Theorem 12. *If $\hat{Y}$ is a Γ-ordered complex and $Y := \Gamma\backslash\hat{Y}$, then $a_i^{(2)}(\hat{Y};\Gamma) \leq \#\mathbb{E}_i^Y$. If, in addition, $\hat{Y}$ is deep-fall, then $a_i^{(2)}(\hat{Y};\Gamma) = \#\mathbb{E}_i^Y$.*

Proof. By Proposition 10,

$$\dim_\Gamma \overline{\partial(\ell^2(\mathbb{I}_i^{\hat{Y}}))} = \dim_\Gamma \overline{\ell^2(\mathbb{I}_i^{\hat{Y}})} = \#\mathbb{I}_i^Y.$$

By the additivity of dimension and Proposition 10,

$$\begin{aligned} a_i^{(2)}(\hat{Y};\Gamma) &= \dim_\Gamma \operatorname{Ker}(\partial\colon \ell^2(\Sigma_i^{\hat{Y}}) \to \ell^2(\Sigma_{i-1}^{\hat{Y}})) \\ &= \dim_\Gamma \ell^2(\Sigma_i^{\hat{Y}}) - \dim_\Gamma \overline{\partial(\ell^2(\Sigma_i^{\hat{Y}}))} \leq \dim_\Gamma \ell^2(\Sigma_i^{\hat{Y}}) - \dim_\Gamma \overline{\partial(\ell^2(\mathbb{I}_i^{\hat{Y}}))} \\ &= \dim_\Gamma \ell^2(\Sigma_i^{\hat{Y}}) - \dim_\Gamma \ell^2(\mathbb{I}_i^{\hat{Y}}) = \#\Sigma_i^Y - \#\mathbb{I}_i^Y = \#\mathbb{E}_i^Y. \end{aligned}$$

If $\hat{Y}$ is deep-fall, then the equality holds by Theorem 11. □

Define the *reduced rank* of a finite graph Y by

$$\bar{r}(Y) := \sum_{K \in Comp(Y)} \max\{0, -\chi(K)\},$$

where $Comp(Y)$ is the set of components of Y. The following was proved in [14, Theorem 14].

Theorem 13. *Let $\hat{Y}$ be a forest with a free cocompact Γ-action and $Y := \Gamma\backslash\hat{Y}$. Then*

$$\bar{r}(Y) = b_1^{(2)}(\hat{Y};\Gamma).$$

5.3. **Submultiplicativity for deep-fall leafages.**

Theorem 14. *Suppose*

- *$\hat{X}$ is a Γ-ordered complex,*
- *$\hat{\alpha} : \hat{Y} \to \hat{X}$ and $\hat{\beta} : \hat{Z} \to \hat{X}$ are Γ-ordered leafages over $\hat{X}$,*
- *$\hat{Y}$ and $\hat{Z}$ are of type $\mathcal{F}$ and deep-fall,*
- *$\hat{S}$ is any product of $\hat{Y}$ and $\hat{Z}$ in the category* $\mathrm{Leaf}(\hat{X},\Gamma)^{\leq}$,
- *$Y := \Gamma\backslash\hat{Y}$, $Z := \Gamma\backslash\hat{Z}$, $S := \Gamma\backslash\hat{S}$.*

Then

$$\#\mathbb{E}_i^S \le \#\mathbb{E}_i^Y \cdot \#\mathbb{E}_i^Z \qquad \textit{and} \qquad a_i^{(2)}(\hat{S};\Gamma) \le a_i^{(2)}(\hat{Y};\Gamma)\cdot a_i^{(2)}(\hat{Z};\Gamma).$$

If, in addition, $\hat{Y}$ and $\hat{Z}$ are forests, then $\hat{S}$ is a forest and $\bar{r}(S) \le \bar{r}(Y)\cdot \bar{r}(Z)$.

Proof. $\hat{S}$ is part of diagram (4). By the definition of fiber product, the map

$$\varphi : \Sigma_i^S \to \Sigma_i^Y \times \Sigma_i^Z, \quad \sigma \mapsto (\mu(\sigma), \nu(\sigma)),$$

is injective. The maps $\hat{\mu} : \hat{S} \to \hat{Y}$ and $\hat{\nu} : \hat{S} \to \hat{Z}$ are morphisms of Γ-leafages over $\hat{X}$. Passing to quotients gives cell maps $\mu : S \to Y$ and $\nu : S \to Z$. By Lemma 9(b),

$$\mu(\mathbb{E}_i^S) \subseteq \mathbb{E}_i^Y \quad \text{and} \quad \nu(\mathbb{E}_i^S) \subseteq \mathbb{E}_i^Z,$$

hence the restriction of φ to $\mathbb{E}_i^S$ takes values in $\mathbb{E}_i^Y \times \mathbb{E}_i^Z$. This implies $\#\mathbb{E}_i^S \le \#\mathbb{E}_i^Y \cdot \#\mathbb{E}_i^Z$. The second inequality follows from Theorem 12:

$$a_i^{(2)}(\hat{S};\Gamma) \le \#\mathbb{E}_i^S \le \#\mathbb{E}_i^Y \cdot \#\mathbb{E}_i^Z = a_i^{(2)}(\hat{Y};\Gamma)\cdot a_i^{(2)}(\hat{Z};\Gamma).$$

The third inequality follows from Theorem 13. □

Remark. With some work it is possible to define $\bar{r}(Y)$ and $a_i^{(2)}(\cdot;\Gamma)$ (and of course $\#\mathbb{E}_i^Y$) without assuming that $\hat{Y}$ is of type $\mathcal{F}$. To do that, one would need to allow infinite values for these numbers. Then it is possible to extend the above theorem to the one without the type $\mathcal{F}$ assumption on $\hat{Y}$ and $\hat{Z}$, by taking limits of type $\mathcal{F}$ complexes. The inequalities hold for infinite values with the convention $0 \cdot \infty = 0$.

For the record, we state the following corollary of Theorem 14.

Theorem 15. *Suppose*

- *$\hat{X}$ is a Γ-ordered complex of type $\mathcal{F}$,*
- *$\hat{X}$ is simply connected (not necessarily connected),*
- *$X := \Gamma\backslash\hat{X}$,*
- *$\alpha : Y \to X$ and $\beta : Z \to X$ are immersions, and*
- *in the Γ-ordered system generated by α, β and p_X, the Γ-ordered complexes $\hat{Y}$ and $\hat{Z}$ are deep-fall.*

Then

$$\#\mathbb{E}_i^S \le \#\mathbb{E}_i^Y \cdot \#\mathbb{E}_i^Z \quad \textit{and} \quad a_i^{(2)}(\hat{S};\Gamma) \le a_i^{(2)}(\hat{Y};\Gamma)\cdot a_i^{(2)}(\hat{Z};\Gamma).$$

If, in addition, $\hat{Y}$ and $\hat{Z}$ are forests, then $\hat{S}$ is a forest and $\bar{r}(S) \le \bar{r}(Y)\cdot \bar{r}(Z)$.

5.4. **The finite-fall property.** Fix $i \ge 0$. A Γ-ordered complex $\hat{Y}$ of type $\mathcal{F}$ will be called *finite-fall*, or more explicitly, *i-finite-fall*, if for any $\sigma \in \mathbb{E}_i^{\hat{Y}}$ and any finite subset $E \subseteq \mathbb{E}_i^{\hat{Y}}$,

$$\partial\sigma \in \overline{\partial(\ell^2([\Sigma_i^{\hat{Y}} < \sigma] \setminus E))}.$$

Lemma 16. *Any Γ-ordered complex $\hat{Y}$ of type $\mathcal{F}$ is finite-fall.*

Proof. Take any $\sigma \in \mathbb{E}_i^{\hat{Y}}$ and any finite $E \subseteq \mathbb{E}_i^{\hat{Y}}$. Without loss of generality we will additionally assume that $E \subseteq [\mathbb{E}_i^{\hat{Y}} < \sigma]$. We prove the statement by induction on the cardinality of E.

If $\#E = 0$, since σ is order-essential,

$$\partial\sigma \in \overline{\partial(\ell^2([\Sigma_i^{\hat{Y}} < \sigma]))} = \overline{\partial(\ell^2([\Sigma_i^{\hat{Y}} < \sigma] \setminus E))}.$$

Now assume that $\#E \geq 1$. Since E is finite, there is an element ω of E that is maximal with respect to the total order on E induced from $\Sigma_i^{\hat{Y}}$. Denote

$$D := [\Sigma_i^{\hat{Y}} < \sigma] \setminus E, \qquad E' := E \setminus \{\omega\},$$

then $[\Sigma_i^{\hat{Y}} < \sigma] \setminus E' = D \cup \{\omega\}$. We have $\omega \in \mathbb{E}_i^{\hat{Y}}$ and $\omega < \sigma$. Since $\#E' < \#E$, the induction hypotheses for ω and E' yield

$$\partial\omega \in \overline{\partial(\ell^2([\Sigma_i^{\hat{Y}} < \omega] \setminus E'))} \subseteq \overline{\partial(\ell^2([\Sigma_i^{\hat{Y}} < \sigma] \setminus E))} = \overline{\partial(\ell^2(D))}.$$

The induction hypotheses for σ and E' yield

$$\begin{aligned}\partial\sigma &\in \overline{\partial(\ell^2([\Sigma_i^{\hat{Y}} < \sigma] \setminus E'))} = \overline{\partial(\ell^2(D \cup \{\omega\}))} \subseteq \overline{\partial(\ell^2(D) + \ell^2(\{\omega\}))} \\ &\subseteq \overline{\partial(\ell^2(D)) + \partial(\ell^2(\{\omega\}))} \subseteq \overline{\partial(\ell^2(D))} = \overline{\partial(\ell^2([\Sigma_i^{\hat{Y}} < \sigma] \setminus E))}\end{aligned}$$

as desired. ☐

6. Graphs.

The main goal of this section is to prove the deep-fall property for graphs.

6.1. **Infinite graphs.** A graph Y will be called *infinite* if it is infinite as a set, i.e. the union of its vertices and edges, $\Sigma_0^Y \sqcup \Sigma_1^Y$, is infinite.

Lemma 17. *Suppose Y is a locally finite graph. Then Y is infinite if and only if its set of vertices Σ_0^Y is infinite.*

Proof. The "if" direction is clear. For the converse, suppose that $\Sigma_0^Y \sqcup \Sigma_1^Y$ is infinite, but Σ_0^Y is finite, then Σ_1^Y is infinite. This is impossible since Y is locally finite. ☐

Lemma 18. *Let Y be a connected graph. Then Y is infinite if and only if its set of edges Σ_1^Y is infinite.*

Proof. The "if" direction is clear. For the converse suppose that Σ_1^Y is finite. Each vertex in Y must be incident with some edge in Σ_1^Y, otherwise it would be isolated in Y, which contradicts connectedness. Hence Y has at most $2 \cdot \#\Sigma_1^Y$ vertices, so Y is not infinite. ☐

6.2. **Relative components and relative graphs.** A connected component of a graph is the set of vertices and edges that can be conected to a given point by a path. We refine this to the notion of a *relative component* as follows.

Let Q be a graph and E be a subset of the edge set Σ_1^Q. An E-*path* in Q, or an (E,Q)-*path*, is a formal finite sequence of vertices and edges of the type $v_0, \sigma_1, v_1, \sigma_2, \ldots$ or of the type $\sigma_1, v_1, \sigma_2, v_2 \ldots$ such that

$$\begin{aligned} &v_i \in \Sigma_0^Q, \quad \sigma_i \in E, \quad \text{and} \\ &(\sigma_i^- = v_i \text{ and } \sigma_i^+ = v_{i-1}) \text{ or } (\sigma_i^- = v_{i-1} \text{ and } \sigma_i^+ = v_i). \end{aligned}$$

(Declare the equalities void when the indices do not make sense.) If $a, b \in Q$ (either vertices or edges), we say that an edge path *connects* a *to* b, if it starts with a and ends with b. In particular, for each vertex $v \in \Sigma_i^Q$, the one-term sequence v is an E-path in Q connecting v to v.

For $E \subseteq \Sigma_1^Q$, denote $gr(E)$ the subgraph of Q generated by E; it is the disjoint union of E together with the vertices of Q that are adjacent to the edges of E.

Definition 19. *Suppose Q is a graph, $E \subseteq \Sigma_1^Q$, and v is a vertex in Q. Define* the component of Q at v relative to E, *or simply* the E-component at v, *denoted $Q(E,v)$, by either of the following equivalent definitions.*

(a) *If $v \in gr(E)$, let $Q(E,v)$ be the connected component of $gr(E)$ containing v. If $v \notin gr(E)$, set $Q(E,v) := \{v\}$.*
(b) *Let $Q(E,v)$ be the set of vertices and edges in Q that can be connected to v by an E-path.*

If Q is a graph and $E \subseteq \Sigma_i^Q$, the relative graph *is the subgraph*

$$Q(E) := \bigcup_{v \in \Sigma_0^Q} Q(E,v) \subseteq Q.$$

We list properties of relative components.

Lemma 20. (1) *$v \in Q(E,v)$. In particular, $Q(E,v)$ is never empty.*
(2) *$Q(E,v)$ is a subgraph of Q.*
(3) *$Q(E,v)$ is connected.*
(4) *If v and w are vertices in Q and $w \in Q(E,v)$, then $Q(E,v) = Q(E,w)$.*
(5) *$Q(E,v)$ is increasing in variable E: if $E \subseteq E'$ then $Q(E,v) \subseteq Q(E',v)$.*
(6) *For any pair of vertices $v, w \in \Sigma_0^Q$,*

$$Q(E,w) \cap Q(E,v) \neq \emptyset \quad \Leftrightarrow \quad Q(E,v) = Q(E,w).$$

Proof. (1)-(5) follow from the definition.

(6) Since relative components are never empty, the direction "$\Leftarrow$" is immediate. For the direction "$\Rightarrow$", since $Q(E,v)$ and $Q(E,w)$ share a vertex or an edge, then v and w can be connected by an E-path, then $Q(E,v) = Q(E,w)$. $\square$

6.3. **Graphs and Hilbert spaces.**

Lemma 21. *Suppose Q is a uniformly locally finite graph, $u, v \in \Sigma_0^Q$, and $E \subseteq \Sigma_1^Q$. Then the following statements are equivalent.*

(1) $v - u \in \overline{\partial(\ell^2(E))}$ *in* $\ell^2(\Sigma_0^Q)$.
(2) *Both* $Q(E,u)$ *and* $Q(E,v)$ *are infinite or* $Q(E,u) \cap Q(E,v) \neq \emptyset$.
(3) *Both* $Q(E,u)$ *and* $Q(E,v)$ *are infinite or* $Q(E,u) = Q(E,v)$.

Proof. $(\mathbf{2}) \Leftrightarrow (\mathbf{3})$ follows from Lemma 20(6).
$(\mathbf{3}) \Rightarrow (\mathbf{1})$. Assume that $Q(E,u)$ and $Q(E,v)$ are infinite. For each n pick a subset $W_n \subseteq \Sigma_0^{Q(E,v)}$ of cardinality n. For each $w \in W_n$ choose an E-path p_w connecting w to v. View p_w as the oriented sum of its edges: the edges oriented in the direction of the path come with coefficient 1, and the others with -1. Then $\partial p_w = v - w$.

$$\left| v - \partial\Big(\frac{1}{n}\sum_{w \in W_n} p_w\Big)\right|_2 = \left| v - \sum_{w \in W_n} \frac{v-w}{n}\right|_2 = \left| \sum_{w \in W_n} \frac{w}{n} \right|_2 = \frac{1}{\sqrt{n}} \to 0 \quad \text{as } n \to \infty,$$

hence $v \in \overline{\partial(\ell^2(E))}$. Similarly, $u \in \overline{\partial(\ell^2(E))}$, hence $v - u \in \overline{\partial(\ell^2(E))}$.

Now assume that $Q(E,u) = Q(E,v)$. There exists an E-path p connecting u to v. View p as the oriented sum of its edges, then $v - u = \partial p \in \overline{\partial(\ell^2(E))}$.
$(\mathbf{1}) \Rightarrow (\mathbf{2})$. Suppose $v - u \in \overline{\partial(\ell^2(E))}$, $Q(E,v)$ is finite and $Q(E,u) \cap Q(E,v) = \emptyset$. Denote $K := Q(E,v)$, then K is a connected component of the relative graph $Q(E) \subseteq Q$ and $u \notin K$. Let $pr_K : \ell^2(Q(E)) \to \ell^2(K)$ be the orthogonal projection. Lemma 7 says that pr_K commutes with $\partial : \ell^2(Q(E)) \to \ell^2(Q(E))$, so

$$v = pr_K(v-u) \in pr_K\Big(\overline{\partial(\ell^2(E))}\Big) \subseteq \overline{pr_K(\partial(\ell^2(E)))} = \overline{\partial(pr_K(\ell^2(E)))} = \overline{\partial(\ell^2(\Sigma_1^K))}.$$

Since Σ_1^K is a finite set, we have

$$v \in \overline{\partial(\ell^2(\Sigma_1^K))} = \overline{\partial(\mathbb{C}\Sigma_1^K)} = \partial(\mathbb{C}\Sigma_1^K).$$

Let $\epsilon : \mathbb{C}\Sigma_0^K \to \mathbb{C}$ be the augmentation map given by

$$\sum_{v \in \Sigma_0^K} \alpha_v v \mapsto \sum_{v \in \Sigma_0^K} \alpha_v.$$

Then $\epsilon(v) = 1$ and $\epsilon(\partial(\mathbb{C}\Sigma_1^K)) = 0$, which is a contradiction. The case when $Q(E,u)$ is finite is done similarly. $\square$

If K is a subgraph of a graph Q, let *the corona* of K in Q be the set

$$Corona(K,Q) := \{\sigma \mid \sigma \text{ is an edge in } Q \setminus K \text{ such that } \sigma^- \in K \text{ or } \sigma^+ \in K\}.$$

For a family $\mathcal{E}$ of subsets of a set Σ, denote

$$\cap\mathcal{E} := \bigcap_{E \in \mathcal{E}} E.$$

The ambient set Σ is part of the structure of $\mathcal{E}$, and in the case $\mathcal{E} = \emptyset$ the above definition formally says that $\cap\emptyset = \Sigma$.

Lemma 22. *Let Q be a locally finite graph, v be a vertex in Q, and $\mathcal{E}$ be a family of subsets in Σ_1^Q. If $Q(\cap\mathcal{E}, v)$ is finite, then there exists a finite subfamily $\mathcal{E}' \subseteq \mathcal{E}$ such that*

$$Q(\cap\mathcal{E}, v) = Q(\cap\mathcal{E}', v).$$

Note. We do allow $\mathcal{E}'$ to be empty. In this case the lemma asserts that

$$Q(\cap\mathcal{E}, v) = Q(\Sigma_1^Q, v).$$

Proof. Take any $\sigma \in Corona(Q(\cap\mathcal{E}, v), Q)$. Then $\sigma \notin Q(\cap\mathcal{E}, v)$ and σ is adjacent to some vertex in $Q(\cap\mathcal{E}, v)$. If we suppose that $\sigma \in \cap\mathcal{E}$, then by the definition of relative components, v can be connected to σ by an $\cap\mathcal{E}$-path, hence $\sigma \in Q(\cap\mathcal{E}, v)$, which is a contradiction. This proves that for each $\sigma \in Corona(Q(\cap\mathcal{E}, v), Q)$, we have $\sigma \notin \cap\mathcal{E}$. Thus for each $\sigma \in Corona(Q(\cap\mathcal{E}, v), Q)$ we can pick some $E_\sigma \in \mathcal{E}$ such that $\sigma \notin E_\sigma$.

Let

$$\mathcal{E}' := \{E_\sigma \mid \sigma \in Corona(Q(\cap\mathcal{E}, v), Q)\}.$$

We have

$$(\cap\mathcal{E}') \cap Corona(Q(\cap\mathcal{E}, v), Q) = \emptyset. \tag{7}$$

Since Q is locally finite and $Q(\cap\mathcal{E}, v)$ is finite, then $Corona(Q(\cap\mathcal{E}, Q), v)$ is finite, so $\mathcal{E}'$ is a finite subfamily of $\mathcal{E}$ (possibly empty). Since $\mathcal{E}' \subseteq \mathcal{E}$, we have $v \in Q(\cap\mathcal{E}, v) \subseteq Q(\cap\mathcal{E}', v)$.

Suppose that the last inclusion is proper. Then there exists a vertex or an edge $a \in Q(\cap\mathcal{E}', v)\setminus Q(\cap\mathcal{E}, v)$. Then there is an $\cap\mathcal{E}'$-path p connecting v to a. By definition, all the edges of p are in $\cap\mathcal{E}'$. Since $a \notin Q(\cap\mathcal{E}, v)$, then the last edge of p is not in $Q(\cap\mathcal{E}, v)$. Among the edges of p, let σ be the first edge that is not in $Q(\cap\mathcal{E}, v)$. All the edges of p that lie before σ are in $Q(\cap\mathcal{E}, v)$, hence in $\cap\mathcal{E}$. Therefore all the edges and vertices of p that lie before σ form an $\cap\mathcal{E}$-path that connects v to the vertex just before σ. Then this vertex lies in $Q(\cap\mathcal{E}, v)$, hence $\sigma \in Corona(Q(\cap\mathcal{E}, v), Q)$. This contradicts (7). The contradiction shows that $Q(\cap\mathcal{E}, v) = Q(\cap\mathcal{E}', v)$. ☐

Lemma 23. *Let Q be a uniformly locally finite graph, σ be an edge in Q, and $\mathcal{E}$ be a family of subsets in Σ_1^Q. Suppose that for any finite subfamily $\mathcal{E}' \subseteq \mathcal{E}$,*

$$\partial\sigma \in \overline{\partial(\ell^2(\cap\mathcal{E}'))} \quad \textit{in} \quad \ell^2(\Sigma_0^Q).$$

Then

$$\partial\sigma \in \overline{\partial(\ell^2(\cap\mathcal{E}))} \quad \textit{in} \quad \ell^2(\Sigma_0^Q).$$

Proof. By Lemma 21, the condition $\partial\sigma \in \overline{\partial(\ell^2(\cap\mathcal{E}'))}$ is equivalent to the statement

$$\begin{aligned} &\text{both } Q(\cap\mathcal{E}', \sigma^-) \text{ and } Q(\cap\mathcal{E}', \sigma^+) \text{ are infinite or} \\ &Q(\cap\mathcal{E}', \sigma^-) = Q(\cap\mathcal{E}', \sigma^+). \end{aligned} \tag{8}$$

We assume that this holds for each finite subfamily $\mathcal{E}' \subseteq \mathcal{E}$. We want to show that $\partial\sigma \in \overline{\partial(\ell^2(\cap\mathcal{E}))}$; this is equivalent to the statement

$$\begin{aligned} &Q(\cap\mathcal{E}, \sigma^-) \text{ and } Q(\cap\mathcal{E}, \sigma^+) \text{ are infinite or} \\ &Q(\cap\mathcal{E}, \sigma^-) = Q(\cap\mathcal{E}, \sigma^+). \end{aligned} \tag{9}$$

Case 1. *Assume that for each finite subfamily $\mathcal{E}' \subseteq \mathcal{E}$, both $Q(\cap\mathcal{E}', \sigma^-)$ and $Q(\cap\mathcal{E}', \sigma^+)$ are infinite.*

Lemma 22 implies that $Q(\cap\mathcal{E}, \sigma^-)$ and $Q(\cap\mathcal{E}, \sigma^+)$ are infinite. This implies (9).

Case 2. *Assume that there exists a finite subfamily $\mathcal{E}' \subseteq \mathcal{E}$ such that $Q(\cap\mathcal{E}', \sigma^-)$ or $Q(\cap\mathcal{E}', \sigma^+)$ is finite.*

For example, $Q(\cap\mathcal{E}', \sigma^-)$ is finite. Since $Q(\cap\mathcal{E}, \sigma^-) \subseteq Q(\cap\mathcal{E}', \sigma^-)$, then $Q(\cap\mathcal{E}, \sigma^-)$ is finite as well. By Lemma 22, there exist a finite subfamily $\mathcal{E}^- \subseteq \mathcal{E}$ such that

$$Q(\cap\mathcal{E}^-, \sigma^-) = Q(\cap\mathcal{E}, \sigma^-).$$

In particular, $Q(\cap\mathcal{E}^-, \sigma^-)$ is finite. Condition (8) applies to the family $\mathcal{E}^-$ and says that

$$Q(\cap\mathcal{E}^-, \sigma^-) = Q(\cap\mathcal{E}^-, \sigma^+).$$

Since $\sigma^+ \in Q(\cap\mathcal{E}^-, \sigma^+)$, the above two equalities imply that $\sigma^+ \in Q(\cap\mathcal{E}, \sigma^-)$. By Lemma 20(5),

$$Q(\cap\mathcal{E}, \sigma^-) = Q(\cap\mathcal{E}, \sigma^+).$$

This implies (9). The same argument goes through under the ussumption that $Q(\cap\mathcal{E}', \sigma^+)$ is finite, by interchanging $+$ and $-$. $\square$

Note that if Q happens to be a forest and $\sigma \notin \cap\mathcal{E}$, the above proof simplifies; it suffices only to deal with Case 1.

Theorem 24 (Deep-fall property for graphs)**.** *Let $\hat{Y}$ be a Γ-ordered graph of type $\mathcal{F}$. Then for any $\sigma \in \mathbb{E}_1^{\hat{Y}}$, $\partial\sigma \in \overline{\partial(\ell^2[\mathbb{I}_1^{\hat{Y}} < \sigma])}$.*

Proof. Take any $\sigma \in \mathbb{E}_1^{\hat{Y}}$. Consider the family

$$\mathcal{E} := \left\{ [\Sigma_1^{\hat{Y}} < \sigma] \setminus \{\eta\} \;\middle|\; \eta \in \mathbb{E}_1^{\hat{Y}} \right\}$$

of subsets in $[\Sigma_1^{\hat{Y}} < \sigma]$. Then $\cap\mathcal{E} = [\Sigma_1^{\hat{Y}} < \sigma] \setminus \mathbb{E}_1^{\hat{Y}} = [\mathbb{I}_1^{\hat{Y}} < \sigma]$.

For each finite subfamily $\mathcal{E}' \subseteq \mathcal{E}$ there exists a finite $E \subseteq \mathbb{E}_1^{\hat{Y}}$ such that

$$\cap\mathcal{E}' = [\Sigma_1^{\hat{Y}} < \sigma] \setminus E.$$

By the finite-fall property (Lemma 16),

$$\partial\sigma \in \overline{\partial(\ell^2([\Sigma_1^{\hat{Y}} < \sigma] \setminus E))} = \overline{\partial(\ell^2(\cap\mathcal{E}'))}.$$

Since $\mathcal{E}'$ is arbitrary, Lemma 23 implies

$$\partial\sigma \in \overline{\partial(\ell^2(\cap\mathcal{E}))} = \overline{\partial(\ell^2[\mathbb{I}_1^{\hat{Y}} < \sigma])}.$$

$\square$

7. The proof of SHNC.

Theorem 25 (The Strengthened Hanna Neumann Conjecture.). *Suppose Γ is a free group and A and B are its finitely generated subgroups. Then*

$$\sum_{z\in s(A\backslash\Gamma/B)} \bar{r}(A^z \cap B) \leq \bar{r}(A)\,\bar{r}(B).$$

Proof. Let $\Gamma := F_2$ and X be a finite graph with $\Gamma \cong \pi_1(X)$. Take immersions of finite graphs $\alpha : Y \to X$ and $\beta : Z \to X$ representing the subgroups $A, B \leq \Gamma$, respectively, as defined by Stallings [21]. Let $p_X : \hat{X} \to X$ be the universal cover.

Free groups are left-orderable, and even two-sided orderable ([20, p.157], [23, p.165]), hence the system generated by α, β and p_X is a Γ-ordered system. By Theorem 24, $\hat{Y}$ and $\hat{Z}$ are deep-fall. Next, one can proceed in two ways.

One way. By Theorem 15, $a_i^{(2)}(\hat{S};\Gamma) \leq a_i^{(2)}(\hat{Y};\Gamma)\cdot a_i^{(2)}(\hat{Z};\Gamma)$. For graphs this is equivalent to

$$b_i^{(2)}(\hat{S};\Gamma) \leq b_i^{(2)}(\hat{Y};\Gamma)\cdot b_i^{(2)}(\hat{Z};\Gamma).$$

By [14, Theorem 25(b')] this is equivalent to SHNC.

Another way. Since $\hat{X}$ is simply connected, then $\hat{\alpha}$ and $\hat{\beta}$ are leafages by [14, Theorem 7(c)]. Since $\hat{X}$ is a tree, then $\hat{Y}$ and $\hat{Z}$ are forests. By Theorem 15, $\bar{r}(S) \leq \bar{r}(Y)\cdot\bar{r}(Z)$. It follows from definitions that $\bar{r}(Y) = \bar{r}(\pi_1(Y)) = \bar{r}(A)$ and $\bar{r}(Z) = \bar{r}(\pi_1(Z)) = \bar{r}(B)$. With some more work, one can check that the sum in the statement of SHNC equals $\bar{r}(S)$. □

Note that theorems 14 and 24 imply the following more general result which does not assume that $\hat{X}$ is simply connected.

Theorem 26. *Suppose*

- *$\hat{X}$ is a Γ-ordered complex,*
- *$\hat{Y}$ and $\hat{Z}$ are Γ-leafages over $\hat{X}$ of type $\mathcal{F}$ (with pull-back orders from $\hat{X}$),*
- *$\hat{Y}$ and $\hat{Z}$ are graphs,*
- *$\hat{S}$ is the product of $\hat{Y}$ and $\hat{Z}$ (with a Γ-invariant pull-back order from $\hat{X}$),*
- *$Y := \Gamma\backslash\hat{Y}$, $Z := \Gamma\backslash\hat{Z}$, $S := \Gamma\backslash\hat{S}$.*

Then

$$\#\mathbb{E}_i^S \leq \#\mathbb{E}_i^Y \cdot \#\mathbb{E}_i^Z \quad \text{and} \quad a_i^{(2)}(\hat{S};\Gamma) \leq a_i^{(2)}(\hat{Y};\Gamma)\cdot a_i^{(2)}(\hat{Z};\Gamma).$$

If, in addition, $\hat{Y}$ and $\hat{Z}$ are forests, then $\hat{S}$ is a forest and $\bar{r}(S) \leq \bar{r}(Y)\cdot\bar{r}(Z)$.

8. Additional remarks.

8.1. **Square maps.** A system is defined in [14] either as diagram (5) or as the diagram

(10)
$$
\mathcal{S}: \quad
\begin{tikzcd}
S \arrow[rrr, "\nu"] \arrow[ddd, "\mu"'] & & & Z \arrow[ddd, "\beta"] & \\
& \hat{S} \arrow[ul, "p_S"'] \arrow[rrr, "\hat{\nu}"] \arrow[dr, "\bar{\iota}"] \arrow[ddd, "\hat{\mu}"'] & & & \hat{Z} \arrow[ul, "p_Z"'] \arrow[dl, "\bar{\beta}"'] \arrow[ddd, "\hat{\beta}"] \\
& & \hat{\hat{S}} \arrow[uull, bend left, "p_Y \square p_Z"] \arrow[urr, "\bar{\nu}"] \arrow[ddl, "\bar{\mu}"'] & & \\
Y \arrow[rrr, "\alpha"] & & & X & \\
& \hat{Y} \arrow[ul, "p_Y"] \arrow[urr, "\bar{\alpha}"] \arrow[rrr, "\hat{\alpha}"] & & & \hat{X}, \arrow[ul, "p_X"']
\end{tikzcd}
$$

where $\hat{\hat{S}}$ is the fiber-product of $\hat{Y}$ and $\hat{Z}$ over X. The two definitions are equivalent since diagrams (5) and (10) determine each other. We used only (5) in this paper, but note that the proofs can be rewritten in terms of $\hat{\hat{S}}$ instead of $\hat{S}$ (that is using the square approach instead of the diagonal approach). In particular, under the assumptions of Theorem 14 it is possible to explicitly describe an injective $\Gamma\times\Gamma$-equivariant *square map* $Z_i^{(2)}(\hat{\hat{S}}) \to Z_i^{(2)}(\hat{Y}) \otimes Z_i^{(2)}(\hat{Z})$, where $Z_i^{(2)}(\hat{Y})$ denotes the kernel of the boundary map in dimension i. The existence of such a map implies SHNC (see [14]).

8.2. **Essential sets of edges.** We can relate the set $\mathbb{E}_1^{\hat{Y}}$ to the following combinatorial notion introduced in [14, subsection 5.3]. A set of edges $E \subseteq \Sigma_1^Y$ in a finite graph Y is called *essential* if $\bar{r}(Y \setminus E) = \bar{r}(Y) - \#E$. A *maximal essential set* is an essential set that is maximal with respect to inclusion. This is equivalent to $Y \setminus E$ being a maximal subgarden of Y as defined in [14, subsection 5.3]. This can also be shown to be equivalent to the condition

$$\bar{r}(Y \setminus E) = 0 = \bar{r}(Y) - \#E.$$

Theorems 12, 13 and 24 imply

Theorem 27. *Let $\hat{Y}$ be a Γ-ordered forest of type $\mathcal{F}$ and $Y := \Gamma\backslash\hat{Y}$. Then*

$$\bar{r}(Y) = b_1^{(2)}(\hat{Y};\Gamma) = a_1^{(2)}(\hat{Y};\Gamma) = \#\mathbb{E}_1^Y.$$

Lemma 28. *Let $\hat{Y}$ be a Γ-ordered forest of type $\mathcal{F}$ and $Y := \Gamma\backslash\hat{Y}$. Then $\mathbb{E}_1^Y$ is a maximal essential set of edges in Y.*

Proof. By Proposition 10 and Theorem 27,

$$\bar{r}(Y \setminus \mathbb{E}_1^Y) = a_1^{(2)}(\hat{Y} \setminus \mathbb{E}_1^{\hat{Y}};\Gamma) = \dim_\Gamma \operatorname{Ker}\left(\partial\colon \ell^2(\mathbb{I}_1^{\hat{Y}}) \to \ell^2(\Sigma_0^{\hat{Y}})\right) = 0 = \bar{r}(Y) - \#\mathbb{E}_1^Y.$$

□

8.3. **The Amalgamated Graph Conjecture.** As an illustration of another face of SHNC we state a purely combinatorial Amalgamated Graph Conjecture (AGC) due to Dicks [2] about bipartite graphs.

A graph P is *bipartite* if its vertex set is the disjoint union of two sets, $V^-(P)$ and $V^+(P)$ (of color $-$ and of color $+$), and each edge goes from a vertex in $V^-(P)$ to a vertex in $V^+(P)$. Maps of bipartite graphs are required to preserve the colors of vertices. *Always additionally require that in a bipartite graph any pair of vertices is connected by at most 1 edge.* Therefore the total number of edges in a bipartite graph P is bounded above by $\#V^-(P) \cdot \#V^+(P)$. We say that a bipartite graph is *at most half-complete* if the number of its edges is bounded above by

$$\frac{\#V^-(P) \cdot \#V^+(P)}{2}.$$

Conjecture (AGC, Dicks [2])**.** *Suppose*

(a) *Δ is a finite bipartite graph,*
(b) *a finite bipartite graph Φ_i is given for each $i \in \mathbb{Z}_3$,*
(c) *an embedding $\Delta \hookrightarrow \Phi_i$ is given for each $i \in \mathbb{Z}_3$,*
(d) *the amalgamation $\Phi_{i-1} \sqcup_\Delta \Phi_{i+1}$ is a bipartite graph (with at most one edge connecting any pair of vertices), and*
(e) *the bipartite graph $\sqcup_i(\Phi_{i-1} \sqcup_\Delta \Phi_{i+1})$ is a disjoint union of two isomorphic bipartite graphs.*

Then Δ is at most half-complete.

Here $\Phi_{i-1} \sqcup_\Delta \Phi_{i+1}$ denotes the quotient of $\Phi_{i-1} \sqcup \Phi_{i+1}$ identifying the images of Δ in Φ_{i-1} and in Φ_{i+1}.

Theorem 29 (Dicks [2])**.** *The Amalgamated Graph Conjecture is equivalent to the Strenghened Hanna Neumann Conjecture.*

Therefore, Theorem 25 implies AGC. SHCN can also be shown to be equivalent to the following combinatorial

Statement. *Suppose*

(a) *P_i are finite bipartite graphs, for $i \in \mathbb{Z}_3$,*
(b) *M, N are finite bipartite graphs,*
(c) *$\varphi : \sqcup_i P_i \to M$ and $\psi : \sqcup_i P_i \to N$ are maps of bipartite graphs whose restrictions to each P_i are injective,*
(d) *for each vertex or edge x in M, $\#\varphi^{-1}(x) = 2$ or 3, and similarly,*
(e) *for each vertex or edge x in N, $\#\psi^{-1}(x) = 2$ or 3.*

Then the bipartite graph

$$\Big(\cap_i \varphi(P_i) \Big) \sqcup \Big(\cap_i \psi(P_i) \Big)$$

is at most half-complete.

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